STOCHASTIC ANALYSIS OF A DISCRETE PARAMETRIC
MARKOV CHAIN MODEL OF A COMPLEX SYSTEM
CONSISTING OF TWO SUB-SYSTEMS

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Abstract
The paper deals with a stochastic model of a complex system composed of two subsystems- A and B. The subsystem-A consists of two identical units in series configuration whereas subsystem-B consists of only a single unit and is capable to keep the system operative when subsystem-A is failed. System failure occurs only when at least one unit of subsystem-A together with subsystem-B is failed. A single repairman is always available with the system to repair a failed unit of subsystem-A and B. Priority in operation as well as in repair is given to the unit of subsystem-A.

Key Words: Transition Probabilities, Regenerative Point, Mean Sojourn Time, Reliability, MTSF, Discrete Parametric Markov Chain.

1. Introduction
Due to the ever increasing demands of society, the systems are becoming complex day by day with modern technology. Several authors including [1,2,4,7] have analyzed complex system models under different model formulation. Gupta and Ramkishan [3] have analyzed a system model which consists of two subsystems A and B connected in series network. Subsystem-A consists of two identical units. Initially one is online and other is kept into cold standby. Subsystem-B consists of only one unit. The system fails completely due to the failure of any of the subsystem-A or subsystem-B. In analyzing the system models by the above authors, the continuous distributions of all the random variables have been considered. Very few authors namely Gupta and Varshney [5,6] have obtained the reliability characteristics of redundant system models under discrete parametric Markov Chain i.e. the random variables denoting failure and repair times follow discrete distributions.

Keeping the above fact in view, the purpose of the present paper is to analyze a complex system model which consists of two subsystems A and B. Initially subsystem-A is operative and subsystem-B is kept into cold standby. Subsystem-A is further composed of two identical units in series configuration whereas subsystem-B consists of only one unit which is dissimilar from the units of subsystem-A. The main point in analyzing the system model under study is that the random variables denoting time to failure and repair are taken of discrete nature having geometric distributions.
The following economic related measures of system effectiveness are obtained by using regenerative point technique:

i) Transition probabilities and mean sojourn times in various states.

ii) Reliability and mean time to system failure.

iii) Point-wise and steady-state availabilities of the system as well as expected up time of the system due to subsystem A and B respectively during time (0, t-1).

iv) Expected busy period of repairman due to subsystem A and B respectively during time (0, t-1).

v) Net expected profit incurred by the system during a finite interval of time and at steady-state.

2. Model Description and Assumptions

i. The system is consisting of two subsystems A and B. Initially subsystem-A is operative and subsystem-B is kept into cold standby.

ii. The subsystem-A has two identical units in series network whereas the subsystem-B has only one unit alone.

iii. Both the units of subsystem-A are identical whereas the unit of subsystem-B is dissimilar from the unit of subsystem-A.

iv. The system failure occurs when at least one unit of subsystem-A and subsystem-B are failed.

v. The priority in operation as well as in repair is given to the unit of subsystem-A over the subsystem-B.

vi. A single repairman is always available with the system to repair a filed unit of both the subsystems.

vii. The random variables denoting the failure times and repair times of the units of subsystem-A and subsystem-B are independent of discrete nature and follow geometric distributions with different parameters.

viii. After repair a unit works as good as new.

3. Notations and States of the System

a) Notations

\[ p_{ij} \] : p.m.f. of failure time of operating unit of subsystem-A and B respectively for i=a, b; \( (p_i + q_i = 1) \).

\[ r_{ij} \] : p.m.f. of repair time of the unit of subsystem-A and B respectively for i=a, b; \( (r_i + s_i = 1) \).

\[ q_{ij} (\cdot), Q_{ij} (\cdot) \] : p.m.f. and c.d.f. of one step or direct transition time from state \( S_i \) to \( S_j \).

\[ p_{ij} \] : Steady state transition probability from state \( S_i \) to \( S_j \).

\[ Z_i (t) \] : Probability that the system sojourns in state \( S_i \) at epochs 0, 1, 2,……., up to (t-1).
\[ \psi_i \] : Mean sojourn time in state \( S_i \).

\[ *, h \] : Symbol and dummy variable used in geometric transform

\[ \text{e.g. } GT \left[ q_{ij}(t) \right] = q_{ij}^*(h) = \sum_{t=0}^{\infty} h^t q_{ij}(t) \]

b) Symbols for the states of the systems

- \( A_{Og} / A_g \) : Unit of subsystem-A in normal (N) mode and operative/good.
- \( B_{OS} / B_S \) : Unit of subsystem-B in normal (N) mode and operative/standby.
- \( A_r / A_{wr} \) : Unit of subsystem-A in total failure (F) mode and under repair/waiting for repair.
- \( B_r / B_{wr} \) : Unit of subsystem-B in total failure (F) mode and under repair/waiting for repair.

With the help of above symbols the possible states of the system along with failure and repair rates are shown in the transition diagram (fig.1):
4. Transition Probabilities

Let $Q_{ij}(t)$ be the probability that the system transits from state $S_i$ to $S_j$ during time interval $(0, t)$ i.e., if $T_{ij}$ is the transition time from state $S_i$ to $S_j$ then

$$Q_{ij}(t) = P[T_{ij} \leq t]$$

By using simple probabilistic arguments, we have

$$Q_{01}(t) = \frac{2p_aq_a}{1-q_a^2} \left(1-\left(q_a^2\right)^{t+1}\right),$$

$$Q_{02}(t) = \frac{p_a^2}{1-q_a^2} \left(1-\left(q_a^2\right)^{t+1}\right),$$

$$Q_{10}(t) = Q_{21}(t) = \frac{r_b q_b}{1-s_a q_b} \left(1-\left(s_a q_b\right)^{t+1}\right),$$

$$Q_{13}(t) = Q_{24}(t) = \frac{r_b p_b}{1-s_a q_b} \left(1-\left(s_a q_b\right)^{t+1}\right),$$

$$Q_{14}(t) = Q_{25}(t) = \frac{s_a p_b}{1-s_a q_b} \left(1-\left(s_a q_b\right)^{t+1}\right),$$

$$Q_{30}(t) = \frac{q_b^2 r_b}{1-q_a^2 s_b} \left(1-\left(q_a^2 s_b\right)^{t+1}\right),$$

$$Q_{31}(t) = \frac{2p_a q_a r_b}{1-q_a^2 s_b} \left(1-\left(q_a^2 s_b\right)^{t+1}\right),$$

$$Q_{32}(t) = \frac{p_a^2 r_b}{1-q_a^2 s_b} \left(1-\left(q_a^2 s_b\right)^{t+1}\right),$$

$$Q_{34}(t) = \frac{2p_a q_a s_b}{1-q_a^2 s_b} \left(1-\left(q_a^2 s_b\right)^{t+1}\right),$$

$$Q_{35}(t) = \frac{p_a s_b^2}{1-q_a^2 s_b} \left(1-\left(q_a^2 s_b\right)^{t+1}\right),$$

$$Q_{43}(t) = Q_{54}(t) = 1-s_a^{t+1}$$

(11)

The steady state transition probabilities from state $S_i$ to $S_j$ can be obtained from (11) by taking $t \to \infty$, as follows:
We observe that the following relations hold:

\[ p_{01} + p_{02} = 1, \quad p_{10} + p_{13} + p_{14} = 1, \]
\[ p_{21} + p_{24} + p_{25} = 1 \]
\[ p_{30} + p_{31} + p_{32} + p_{34} + p_{35} = 1, \quad p_{43} = p_{54} = 1 \]

(12–16)

5. **Mean Sojourn Times**

\( \psi_i \) be the sojourn time in state \( S_i \) \( (i = 0, 1, 2, 3, 4, 5) \) then mean sojourn time in state \( S_i \) is given by

\[ \psi_i = \sum_{t=1}^{\infty} P[T \geq t] \]

In particular,

\[ \psi_0 = \frac{q_a^2}{p_a}, \quad \psi_1 = \psi_2 = \frac{s_a q_b}{(1-s_a q_b)} = \psi, \]
\[ \psi_3 = \frac{q_a^2 s_b}{(1-q_a^2 s_b)} \]
\[ \psi_4 = \psi_5 = \frac{s_a}{r_a} = \psi_A \]

(17–20)
6. Methodology for Developing Equations

In order to obtain various interesting measures of system effectiveness we develop the recurrence relations for reliability, availability, busy period of repairman as follows:

a) Reliability of the system

Here we define $R_i(t)$ as the probability that the system does not fail up to epochs 0, 1, 2,..., (t-1) when it is initially started from up state $S_i$. To determine it, we regard the failed states $S_4$ and $S_5$ as absorbing states. Now, the expressions for $R_i(t); i=0, 1, 2, 3;$ we have the following set of convolution equations.

$$R_0(t) = q_{AA}^2 + \sum_{u=0}^{t-1} q_{01}(u) R_1(t-1-u) + \sum_{u=0}^{t-1} q_{02}(u) R_2(t-1-u)$$

$$= Z_0(t) + q_{01}(t-1) \odot R_1(t-1) + q_{02}(t-1) \odot R_2(t-1)$$

Similarly,

$$R_1(t) = Z(t) + q_{10}(t-1) \odot R_0(t-1) + q_{12}(t-1) \odot R_2(t-1) + q_{13}(t-1) \odot R_3(t-1)$$

$$R_2(t) = Z(t) + q_{21}(t-1) \odot R_1(t-1)$$

$$R_3(t) = Z(t) + q_{10}(t-1) \odot R_0(t-1) + q_{31}(t-1) \odot R_1(t-1) + q_{12}(t-1) \odot R_2(t-1)$$

(21-24)

Where,

$$Z(t) = s_a^i q_b^i , \quad Z_3(t) = q_b^i s_b^i$$

b) Availability of the system-

Let $A^a_i(t)$ and $A^b_i(t)$ be the respective probabilities that the system is up at epoch t-1 due to operation of unit of subsystem A and B, when it initially started from state $S_i$. Then, by using simple probabilistic arguments as in case of reliability, the following recurrence relations can be easily developed for $A^a_i(t)$ and $A^b_i(t); i=0$ to 5.

$$A^a_0(t) = (1-\delta) Z_0(t) + q_{01}(t-1) \odot A^a_1(t-1) + q_{02}(t-1) \odot A^a_2(t-1)$$

$$A^a_1(t) = \delta Z(t) + q_{10}(t-1) \odot A^a_0(t-1) + q_{12}(t-1) \odot A^a_2(t-1) + q_{13}(t-1) \odot A^a_3(t-1)$$

$$A^a_2(t) = \delta Z(t) + q_{21}(t-1) \odot A^a_1(t-1) + q_{24}(t-1) \odot A^a_4(t-1) + q_{25}(t-1) \odot A^a_5(t-1)$$

$$A^b_0(t) = (1-\delta) Z_0(t) + q_{01}(t-1) \odot A^b_1(t-1) + q_{02}(t-1) \odot A^b_2(t-1)$$

$$A^b_1(t) = \delta Z(t) + q_{10}(t-1) \odot A^b_0(t-1) + q_{12}(t-1) \odot A^b_2(t-1) + q_{13}(t-1) \odot A^b_3(t-1)$$

$$A^b_2(t) = \delta Z(t) + q_{21}(t-1) \odot A^b_1(t-1) + q_{24}(t-1) \odot A^b_4(t-1) + q_{25}(t-1) \odot A^b_5(t-1)$$
\[ A_j^1(t) = (1-\delta)Z_3 + q_{30}(t-1)A_0^1(t-1) + q_{31}(t-1)A_1^1(t-1) + q_{32}(t-1)A_2^1(t-1) + q_{34}(t-1)A_4^1(t-1) + q_{35}(t-1)A_5^1(t-1) \]

\[ A_j^2(t) = q_{43}(t-1)A_3^1(t-1) \]

\[ A_j^3(t) = q_{54}(t-1)A_4^1(t-1) \]

(25-30)

Where,

\[ \delta = 0 \text{ and } 1 \text{ respectively for } j=a, b. \]

The values of \( Z(t) \) and \( Z_j(t); i=0 \) and 3 are same as given in section 6(a).

c) Busy period of repairman

Let \( B_i^a(t) \) and \( B_i^b(t) \) be the respective probabilities that the repairman is busy at epoch (t-1) in the repair of failed unit of subsystem A and B when system initially starts from state \( S_i \). Using simple probabilistic arguments in case of reliability and availability analysis, the relations for \( B_i^j(t); i=0 \) to 5 can be easily developed as below.

\[ B_0^1(t) = q_{01}(t-1)B_1^1(t-1) + q_{01}(t-1)B_2^1(t-1) \]

\[ B_1^1(t) = (1-\delta)Z(t) + q_{10}(t-1)B_0^1(t-1) + q_{13}(t-1)B_3^1(t-1) + q_{14}(t-1)B_4^1(t-1) \]

\[ B_2^1(t) = (1-\delta)Z + q_{21}(t-1)B_1^1(t-1) + q_{24}(t-1)B_4^1(t-1) + q_{25}(t-1)B_5^1(t-1) \]

\[ B_3^1(t) = \delta Z_3(t) + q_{30}(t-1)B_0^1(t-1) + q_{31}(t-1)B_1^1(t-1) + q_{32}(t-1)B_2^1(t-1) + q_{34}(t-1)B_4^1(t-1) + q_{35}(t-1)B_5^1(t-1) \]

\[ B_4^1(t) = Z'(t) + q_{43}(t-1)B_3^1(t-1) \]

\[ B_5^1(t) = Z'(t) + q_{54}(t-1)B_4^1(t-1) \]

(31-36)

Where, \( \delta = 0 \) and 1 respectively for \( j=a, b. \) The values of \( Z(t) \) and \( Z_j(t); i=0, 3 \) are same as given in section 6(a) and \( Z'(t) = S_a^i. \)

7. Analysis of Reliability and MTSF

Taking geometric transforms of (21-24) and simplifying the resulting set of algebraic equations for \( R_0^*(h) \) we get

\[ R_0^*(h) = \frac{N_1(h)}{D_1(h)} \]

(37)

Where,
\[ N_1(h) = \left[1 - hq_{i_3}^* \left( hq_{q_{21}} + h^2 q_{q_{21}}^q q_{21} \right) \right] Z_0^* + \left[ hq_{i_3}^* \left( 1 + h^2 q_{i_{3}q_{32}}^* \right) + hq_{i_2}^* \left( 1 + h^2 q_{q_{12}}^* - h^2 q_{q_{13}}^* q_{31} \right) \right] Z^* + \]
\[ + hq_{i_3}^* \left[ hq_{q_{31}} + h^2 q_{q_{32}}^q q_{21} \right] Z^* \]
\[ D_i(h) = 1 - hq_{i_3}^* \left( hq_{q_{31}} + h^2 q_{q_{32}}^q q_{21} \right) - \left( hq_{1r_0}^* + h^2 q_{i_{3}q_{32}}^* \right) \left( hq_{i_1}^* + h^2 q_{i_{2}q_{32}}^* q_{21} \right) \]

Collecting the coefficient of \( h^i \) from expression (37), we can get the reliability of the system \( R_0(t) \). The MTSF is given by-

\[ E(T) = \lim_{h \to 1} \sum_{i=1}^{\infty} h^i R(t) = \frac{N_1(1)}{D_1(1)} - 1 \] (38)

\[ N_1(1) = \left[1 - p_{13} + p_{32} + p_{31} \right] \psi_0 + \left[ 1 + p_{i0} p_{13} p_{32} + p_{i2} \left( p_{21} - p_{i3} p_{31} \right) \right] \psi_1 + \left[ p_{i3} \left( p_{i0} + p_{i2} p_{21} \right) \right] \psi_2 \]
\[ D_1(1) = 1 - p_{13} + p_{32} p_{21} - \left( p_{i0} + p_{i3} p_{30} \right) \left( p_{01} + p_{02} p_{21} \right) \]

8. Availability Analysis

On taking geometric transforms of (25-30) and simplifying the resulting equations for \( A^a_i(t) \) and \( A^b_i(t) \) we get

\[ A^a_0(h) = \frac{N_2(h)}{D_2(h)} \quad \text{and} \quad A^b_0(h) = \frac{N_3(h)}{D_2(h)} \] (39-40)

Where,

\[ N_2(h) = \left[1 - hq_{i_3}^* \left( hq_{q_{31}} + h^2 q_{q_{21}}^q q_{21} \right) - h^2 q_{q_{21}^q q_{32}} \left( hq_{q_{24}} + h^2 q_{q_{23}}^q q_{54} \right) - \left( h^2 q_{q_{12}}^q q_{21} + hq_{31}^* \right) \left( hq_{i_3} + h^2 q_{i_4 q_{32}}^q \right) \right] Z_0^* + \left[ hq_{i_3}^* \left( 1 + h^2 q_{i_{3}q_{32}}^* \right) + hq_{i_2}^* \left( 1 + h^2 q_{q_{12}}^* - h^2 q_{q_{13}}^* q_{31} \right) \right] Z^* + \]
\[ + hq_{q_{21}}^* \left[ hq_{q_{31}} + h^2 q_{q_{32}}^q q_{21} \right] Z^* \]
\[ N_3(h) = \left[ hq_{i_3}^* \left\{ 1 - hq_{q_{43}} \left( hq_{q_{34}} + h^2 q_{35} q_{54} \right) - h^2 q_{q_{43} q_{32}} \left( hq_{q_{24}} + h^2 q_{55} q_{54} \right) + hq_{q_{32}} \left( hq_{i_3} + h^2 q_{i_4 q_{43}}^q \right) \right\} \right] + hq_{i_2}^* \left\{ \left( 1 + hq_{q_{21}} \right) \left( 1 - hq_{q_{43}} \left( hq_{q_{34}} + h^2 q_{55} q_{54} \right) + hq_{q_{32}} \left( hq_{i_3} + h^2 q_{i_4 q_{43}}^q \right) \right) \right\} \]
\[ D_2(h) = 1 - hq_{q_{43}} \left( hq_{q_{34}} + h^2 q_{35} q_{54} \right) - h^2 q_{q_{43} q_{32}} \left( hq_{q_{24}} + h^2 q_{55} q_{54} \right) - \left( h^2 q_{q_{13}} + hq_{31}^* \right) \left( hq_{i_3} + h^2 q_{i_4 q_{43}}^q \right) \]
\[ \quad + hq_{i_3}^* \left\{ 1 - hq_{q_{43}} \left( hq_{q_{34}} + h^2 q_{35} q_{54} \right) - h^2 q_{q_{43} q_{32}} \left( hq_{q_{24}} + h^2 q_{55} q_{54} \right) + hq_{q_{32}} \left( hq_{i_3} + h^2 q_{i_4 q_{43}}^q \right) \right\} \]
\[ \quad + hq_{i_2}^* \left\{ hq_{q_{21}} \left( 1 - hq_{q_{43}} \left( hq_{q_{34}} + h^2 q_{35} q_{54} \right) + h^2 q_{q_{43} q_{32}} \left( hq_{q_{24}} + h^2 q_{55} q_{54} \right) + hq_{q_{32}} \left( hq_{i_3} + h^2 q_{i_4 q_{43}}^q \right) \right) \right\} + hq_{i_3}^* \left\{ hq_{q_{21}} \left( 1 - hq_{q_{43}} \left( hq_{q_{34}} + h^2 q_{35} q_{54} \right) + h^2 q_{q_{43} q_{32}} \left( hq_{q_{24}} + h^2 q_{55} q_{54} \right) + hq_{q_{32}} \left( hq_{i_3} + h^2 q_{i_4 q_{43}}^q \right) \right) \right\} \]
\[ \quad + hq_{q_{21}} \left( hq_{q_{31}} + h^2 q_{i_4 q_{43}}^q \right) \]
The steady state availabilities of the system due to the operation of unit of subsystem $A$ and unit $B$ of the system are given by

$$A^a_0 = \lim_{t \to \infty} A^a_0(t) = \lim_{h \to 1} (1-h) \frac{N_2(h)}{D_2(h)}$$

$$A^b_0 = \lim_{t \to \infty} A^b_0(t) = \lim_{h \to 1} (1-h) \frac{N_3(h)}{D_2(h)}$$

But $D_2(h)$ at $h=1$ is zero, therefore by applying L' Hospital rule, we get

$$A^a_0 = -\frac{N_2(1)}{D'_2(1)} \quad \text{and} \quad A^b_0 = -\frac{N_3(1)}{D'_2(1)} \quad (41-42)$$

Where,

$$N_2(1) = \left[ p_{10} \left( p_{31} + p_{32}p_{21} \right) + p_{24} \right] \psi_0 + \left[ 1 - p_{10} \left( p_{01} + p_{02}p_{21} \right) \right] \psi_3$$

$$N_3(1) = \left[ 1 - p_{34} - p_{35} - p_{10} \left( p_{01}p_{32} - p_{02}p_{31} \right) + p_{21} \left( p_{32} + p_{30}p_{02} \right) \right] \psi$$

and

$$D'_2(1) = -\left[ \left\{ p_{10} \left( p_{31} + p_{32}p_{21} \right) + p_{24} \right\} \psi_0 + \left\{ 1 - p_{34} - p_{35} - p_{10} \left( p_{01}p_{32} - p_{02}p_{31} \right) + p_{21} \left( p_{32} + p_{30}p_{02} \right) \right\} \psi_3 + \left\{ 1 - p_{10} \left( p_{01} + p_{02}p_{21} \right) \right\} \psi_5 + \left\{ 1 - p_{10}p_{01} - p_{13}p_{33}p_{21} - p_{02}p_{21} \left( p_{10} + p_{13}p_{30} \right) - p_{13} \left( p_{31} + p_{30}p_{01} \right) + \left( 1 - p_{10}p_{01} \right) \left( p_{35} + p_{32}p_{25} \right) + p_{02} \left( p_{25} \left( p_{30} + p_{33}p_{10} \right) - p_{21}p_{10}p_{13} \right) \right\} \psi_A \right]$$

Now the expected uptime of the system due to the working of subsystem $A$ and $B$ respectively upto epoch $(t-1)$ are given by

$$\mu^a_{up}(t) = \sum_{x=0}^{t-1} A^a_0(x) \quad \text{and} \quad \mu^a_{up}(t) = \sum_{x=0}^{t-1} A^a_0(x)$$

so that

$$\mu^a_{up}(h) = \frac{A^a_0(h)}{1-h} \quad \text{and} \quad \mu^b_{up}(h) = \frac{A^b_0(h)}{1-h} \quad (43-44)$$

9. Busy Period Analysis

On taking geometric transforms of (31-36) and simplifying the resulting equations for $j=a$ and $b$ we get

$$B^a_0(h) = \frac{N_4(h)}{D_2(h)} \quad \text{and} \quad B^b_0(h) = \frac{N_5(h)}{D_2(h)} \quad (45-46)$$

Where,

$$N_4(h) = \left[ hq_{01} \left\{ 1 - hq_{43} \left( hq_{34} + h^2q_{35}q_{54} \right) - h^2q_{43}q_{32} \left( hq_{24} + h^2q_{25}q_{54} \right) + hq_{32} \left( hq_{13} + h^2q_{14}q_{43} \right) \right\} \right] Z^*$$

$$N_5(h) = \left[ hq_{01} \left\{ 1 - hq_{43} \left( hq_{34} + h^2q_{35}q_{54} \right) - h^2q_{43}q_{32} \left( hq_{24} + h^2q_{25}q_{54} \right) + hq_{32} \left( hq_{13} + h^2q_{14}q_{43} \right) \right\} \right] Z^*$$
N_5(h) = \left[ hq_{01}^* \left( hq_{13}^* + h^2 q_{14}^* q_{43}^* \right) + hq_{02}^* \left( hq_{24}^* + h^2 q_{25}^* q_{54}^* \right) + hq_{21}^* \left( hq_{13}^* + h^2 q_{14}^* q_{43}^* \right) \right] Z_3^* \\
and D_2 (h) is same as in availability analysis.

In the long run the respective probabilities that the repairman is busy in the repair of failed unit of subsystem A and B are given by:

B_0^a = \lim_{t \to \infty} B_o^a (t) = \lim_{h \to 1} \left( 1 - h \right) \frac{N_4(h)}{D_2(h)}

B_0^b = \lim_{t \to \infty} B_o^b (t) = \lim_{h \to 1} \left( 1 - h \right) \frac{N_5(h)}{D_2(h)}

But D_2(h) at h=1 is zero, therefore by applying L. Hospital rule, we get

B_0^a = - \frac{N_4(1)}{D_2'(1)} and B_0^b = - \frac{N_5(1)}{D_2'(1)} (47-48)

Where,

N_4(1) = \left[ 1 - p_{34} - p_{35} - p_{10} \left( p_{01} p_{32} - p_{02} p_{31} \right) + p_{21} \left( p_{32} + p_{33} p_{21} \\
- p_{02} p_{21} \left( p_{10} + p_{13} p_{30} \right) - p_{13} \left( p_{31} + p_{30} p_{01} \right) + \left( 1 - p_{10} p_{01} \right) \left( p_{35} + p_{32} p_{25} \right) \\
+ p_{02} \left( p_{25} \left( p_{30} + p_{31} p_{10} \right) - p_{21} p_{10} p_{35} \right) \right] \Psi_A \\
N_5(1) = \left[ 1 - p_{10} \left( p_{01} + p_{02} p_{21} \right) \right] \Psi_A

and D_2'(1) is same as in availability analysis.

Now the expected busy period of the repairman in repair of failed unit of subsystem-A and B up to epoch (t-1) are respectively given by:

\mu_b^a (t) = \sum_{x=0}^{t-1} B_o^a (x) \\
\mu_b^b (t) = \sum_{x=0}^{t-1} B_o^b (x)

So that,

\mu_b^{a*} (h) = \frac{B_o^{a*} (h)}{(1 - h)} , \quad \mu_b^{b*} (h) = \frac{B_o^{b*} (h)}{(1 - h)} (49-50)

10. Profit Function Analysis

We are now in the position to obtain the net expected profit incurred up to epoch (t-1) by considering the characteristics obtained in earlier section.

Let us consider,

K_0 = revenue per-unit time by the system when it is operative due to unit of subsystem-A.

K_1 = revenue per-unit time by the system when it is operative due to unit of subsystem-B.

K_2 = cost per-unit time when repairman is busy in the repairing failed unit of subsystem-A.

K_3 = cost per-unit time when repairman is busy in the repairing failed unit of subsystem-B.
Then, the net expected profit incurred up to epoch (t-1) given by
\[ P(t) = K_0\mu^a_{up}(t) + K_1\mu^b_{up}(t) - K_2\mu^a_{b}(t) - K_3\mu^b_{b}(t) \]  
(51)

The expected profit per unit time in steady state is given by-
\[ P = \lim_{t \to \infty} \frac{P(t)}{t} = \lim_{h \to 1} (1-h)^2 P^*(h) \]
\[ = K_0 \lim_{h \to 1} (1-h)^2 \frac{A^a_0(h)}{(1-h)} + K_1 \lim_{h \to 1} (1-h)^2 \frac{A^b_0(h)}{(1-h)} - K_2 \lim_{h \to 1} (1-h)^2 \frac{B^a_0(h)}{(1-h)} - K_3 \lim_{h \to 1} (1-h)^2 \frac{B^b_0(h)}{(1-h)} \]
\[ = K_0 A^a_0 + K_1 A^b_0 - K_2 B^a_0 - K_3 B^b_0 \]  
(52)

11. Graphical Representation

The curves for MTSF and profit function have been drawn for different values of parameters. Fig. 2 depicts the variations in MTSF with respect to failure rate \( p_a \) of operative unit of subsystem-A for different values of repair rate \( r_a \) of failed unit of subsystem-A and failure rate \( p_b \) of operative unit of subsystem-B when repair rate \( r_b \) of unit of subsystem-B is kept fixed as \( r_b = 0.35 \). The smooth curves show the trends for three different values 0.4, 0.6 and 0.8 of \( r_a \) when \( p_b \) is taken as 0.17 where as dotted curves shows the trend for same three values of \( r_a \) as above when \( p_b \) is taken as 0.30 . From the curves we observed that MTSF decreases uniformly as the value of \( p_a \) and \( p_b \) increase and increase with the increase in \( r_a \).

Similarly, Fig. 3 reveals the variation in profit (P) with respect to \( p_a \) for varying values of \( r_a \) and \( p_b \) as in case of MTSF, when the values of other parameters are kept fixed as \( r_b = 0.35 \), \( K_0 = 120 \), \( K_1 = 80 \), \( K_2 = 420 \) and \( K_3 = 300 \). From this figure same trends in respect of \( p_a \), \( r_a \) and \( p_a \) have been observed as in MTSF. Further it is also revealed smooth curves that system is profitable only if \( p_a \) is less than 0.221, 0.290 and 0.387 respectively for \( r_a = 0.4 \), 0.6 and 0.8 for fixed \( p_b = 0.17 \). From dotted curves that system is profitable only if \( p_a \) is less than 0.215, 0.280 and 0.365 respectively for \( r_a = 0.4 \), 0.6 and 0.8 for fixed \( p_b = 0.30 \).
12. References


Fig. 2: Behavior of MTSF with respect to $p_A$, $r_A$ and $p_B$

Fig. 3: Behavior of Profit ($P$) with respect to $p_A$, $r_A$ and $p_B$