A STUDY OF LINEAR COMBINATION BASED FACTOR-TYPE ESTIMATOR FOR ESTIMATING POPULATION MEAN IN SAMPLE SURVEYS

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Abstract
The objective of this paper is to study the linear combination of factor-type (F-T) estimator to estimate population mean and its properties like bias, mean squared error (m.s.e.) etc. along with numerical study over different populations. The expressions of bias and mean squared error (m.s.e.) of the estimator are derived in the form of population parameters by using the concept of large sample approximations. The combination of F-T estimator is compared with some existing estimators and found better over existing estimators in case of negative correlation between study and auxiliary variables or equal efficient and tested by empirical study performed over several data sets.

Key Words: Linear Combination, Factor Type Estimator, Bias, Mean Squared Error.

1. Introduction:
The problem of estimation of population parameters is a burning issue in sample surveys and a number of methods have been advocated to improve the efficiency of estimators. In sample surveys, auxiliary information is used to select the units in the sample and for estimation of the population parameter, to improve the efficiency of the estimators. The ratio, product and regression techniques are used without any cause in sample surveys.

Cochran (2005) advocated on the use of additional information at estimation stage and discussed on ratio type estimator as well while coefficient of correlation between study variable and auxiliary variable is highly positive. Murthy (1964) presented product type estimator to estimate population mean while the study variable and auxiliary variable negatively correlated, Sisodia and Dwivedi (1981) utilized coefficient of variation of auxiliary variate. A dual to ratio estimator proposed by Srivenkataramana (1980), Singh and Tailor (2005) and Tailor and Sharma (2009) worked on ratio-cum-product estimators. Some other valuable contributions are Steel and Torrie (1960), Singh and Kumar (2011), Singh et al. (2012), Sanaullah et al. (2012) etc.

Singh and Shukla (1987) have discussed a family of factor-type (F-T) ratio estimator for estimating population mean. In continuation, one more contribution by Singh and Shukla (1993) is an efficient-factor-type (F-T) estimator for estimating the population mean. Singh and Agnihotri (2008) studied on a general class of estimators using auxiliary information in sample surveys and in the same they considered a linear combination of two estimators and proved the suggested one is better than other
existing estimators. Sharma and Tailor (2010) considered a linear combination of ratio and dual to ratio estimator and proved that the ratio-cum-dual to ratio estimator of finite population mean in simple random sampling is better than some other existing estimators. Deriving motivation from all these evidences, we are studying on a linear combination based factor-type (F-T) estimator to estimate population mean.

Let a simple random sample of size \( n \) is drawn from a finite population \( U = \{1, 2, 3, \ldots, N\} \) of size \( N \). Let \((y_i, x_i)\) be the pairs of observations, where \( y_i \) is the value of variable \( Y \) under consideration and \( x_i \) is the value of auxiliary variable \( X \) of \( i^{th} \) individual.

The motivation is derived from Singh and Shukla (1987) and they have discussed a family of factor type ratio estimator for estimating population mean. The factor type estimator is given by Singh and Shukla (1987) as

\[
\bar{y}_{FT} = \bar{y} \left( \frac{A + C \bar{X} + fB \bar{X}}{A + B \bar{X} + C \bar{X}} \right)
\]

...(1.1)

where, \( A = (k - 1)(k - 2) \); \( B = (k - 1)(k - 4) \); \( C = (k - 2)(k - 3)(k - 4) \); \( 0 \leq k \leq \infty \); \( f = n/N \)

and \( \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \), \( \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \) being the sample mean of study variable and auxiliary variable respectively.

For the study and comparison of any estimator we usually obtain the expressions of bias and mean squared error (m.s.e.) of estimator in the form of parameters and for this purpose the concept of large sample approximations used. Under this, for a large sample, consider \( \bar{y} = \bar{y}(1 + e_0) \) and \( \bar{x} = \bar{x}(1 + e_1) \), where, \(|e_0| < 1 \) and \(|e_1| < 1 \) are error terms. Hence, we have \( E(e_0) = E(e_1) = 0 \), \( E(e_0^2) = \left( \frac{1}{n} - \frac{1}{N} \right) C_y \), \( E(e_1^2) = \left( \frac{1}{n} - \frac{1}{N} \right) C_x \) and \( E(e_0 e_1) = \left( \frac{1}{n} - \frac{1}{N} \right) \rho C_y C_x . \)

By using the concept of large sample approximations the expressions of bias and mean squared error (m.s.e.) of the factor type (F-T) estimator \( \bar{y}_{FT} \) in the form of population parameters are given by:

\[
B(\bar{y}_{FT}) = \bar{y} \left( \bar{y} - \theta \right) \left( \frac{1 - f}{n} \right) \left( \theta C_x - \rho C_y C_x \right)
\]

...(1.2)

\[
M(\bar{y}_{FT}) = \bar{y} \left( \frac{1 - f}{n} \right) \left[ C_y^2 + \theta^2 C_x^2 + 2 \theta \rho C_x C_y \right]
\]

...(1.3)

respectively and the equation of minimum mean squared error of \( \bar{y}_{FT} \) at \( \theta = \theta_0 - \theta_1 = \rho C_x / C_y = -V \) (say) is given by:

\[
M(\bar{y}_{FT})_{\text{min}} = \left( \frac{1 - f}{n} \right) \left[ 1 - \rho^2 \right] S_y^2
\]

...(1.4)

Note that on simplifying the relation \( \theta = -V \), we have the cubic equation in the form of \( k \) which is given by:

\[
(V + 1) k^3 + (fV + f - 8V + 9) k^2 - (5fV + 5f - 23V + 26) k + (4fV + 4f - 22V + 24) = 0
\]

...(1.5)
By solving this expression one can get at most three values of \( k \) like \( k_1, k_2 \) and \( k_3 \) for which m.s.e. is optimal.

**Remark 1:** Define,

\[
\begin{align*}
\theta &= \theta_c - \theta_r, \quad \theta_1 = \frac{fB}{A + fB + C}, \quad \theta_2 = \frac{C}{A + fB + C}, \quad \theta_3 = \frac{A + C}{A + fB + C}, \quad \theta_4 = \frac{A + fB}{A + fB + C}, \quad C_r = \frac{S_r}{Y}, \\
C_r &= \frac{S_r}{X}, \quad V = \rho \frac{C_r}{C}, \quad f = \frac{n}{N}, \quad s_i^2 = \frac{1}{N-1} \sum (y_i - \bar{Y})^2, \quad s_y^2 = \frac{1}{N-1} \sum (y_i - \bar{Y})^2 \quad \text{and} \quad \rho_{xy} = \text{correlation coefficient between } X \text{ and } Y.
\end{align*}
\]

Also, \( \theta_1 + \theta_2 + \theta_3 + \theta_4 = -(\theta_r - \theta_c) \).

2. **The Linear Combination Based F-T Estimator**

Under the same situation as discussed before we consider linear combination of factor-type estimator as follows:

\[
\bar{y}_{FTRP} = f\phi(k) + (1-f)\psi(k) \tag{2.1}
\]

where,

\[
\phi(k) = \frac{y}{x} \left( \frac{A + C}{A + fB} x + \frac{fB}{C} x \right); \quad \psi(k) = \frac{y}{x} \left( \frac{(A + C) x + fB x}{(A + fB) x + C x} \right)
\]

\[ A = (k-1)(k-2); \quad B = (k-1)(k-4); \quad C = (k-2)(k-3)(k-4); \quad \text{if } 0 \leq k \leq \infty \]

2.1 **Some Special Cases**

The estimator \( \bar{y}_{FTRP} \) provides linear combination of some existing estimators on some special values of \( k \) as shown in table 1.

**Table 1: The Estimator \( \bar{y}_{FTRP} \) in Special Cases**

<table>
<thead>
<tr>
<th>Value of ( k )</th>
<th>Estimator</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = 1 )</td>
<td>( \bar{y}_{FTRP} ) = ( f \frac{\bar{X}}{X} + (1-f)\bar{X} )</td>
<td>Linear combination of Ratio-Product type estimator.</td>
</tr>
<tr>
<td>( k = 2 )</td>
<td>( \bar{y}_{FTRP} ) = ( f \frac{\bar{X}}{X} + (1-f)\bar{X} )</td>
<td>Linear combination of Product-Ratio type estimator.</td>
</tr>
<tr>
<td>( k = 3 )</td>
<td>( \bar{y}_{FTRP} ) = ( f \frac{\bar{Y} - f\bar{X}}{(1-f)\bar{X}} + \frac{\bar{Y} - f\bar{X}}{\bar{X}} )</td>
<td>Linear combination of Dual To Ratio-Product type estimator.</td>
</tr>
<tr>
<td>( k = 4 )</td>
<td>( \bar{y}_{FTRP} ) = ( \bar{Y} )</td>
<td>Unbiased estimator.</td>
</tr>
</tbody>
</table>

It is clear that from the table 1, \( \bar{y}_{FTRP} \) provides the linear combination of ratio and product type estimator at \( k = 1 \) and at \( k = 2 \). Also, at \( k = 3 \) it provides the linear combination of dual to ratio estimator and at \( k = 4 \) suggested one is unbiased estimator.
3. Properties of the Estimator $\bar{y}_{FTRP}$

In this section, we shall focus on the bias and mean squared error (m.s.e.) of the estimator $\bar{y}_{FTRP}$ along with its minimum m.s.e. at its optimum value of $k$. The estimator $\bar{y}_{FTRP}$ could be written in terms of $e_i$ and $e_j$ as

$$\bar{y}[e_i + (\theta_1 - \theta_2) + (2f - 1)e_j] = \{2f - 1)e_i + (\theta_1 - \theta_2)e_j\} \quad \ldots(3.1)$$

and the estimator $\bar{y}_{FTRP}$ is found to be biased and it is in the form of parameters is:

$$B(\bar{y}_{FTRP}) = \bar{y}(\theta_1 - \theta_2) \left(1 - \frac{1}{N}\right) \left\{\theta_1 - f\theta_2 + (2f - 1)\theta_2\right\} \quad \ldots(3.2)$$

Also, the m.s.e. of $\bar{y}_{FTRP}$ is:

$$M(\bar{y}_{FTRP}) = \bar{y} \left(1 - \frac{1}{N}\right) \left[C_r^2 + (\theta_1 - \theta_2)^2(2f - 1)C_x^2 + 2(\theta_1 - \theta_2)(2f - 1)pC_xC_r\right] \quad \ldots(3.3)$$

and minimum m.s.e. of $\bar{y}_{FTRP}$ at $\theta_1 - \theta_2 = -\frac{V}{2f - 1}$ is

$$M(\bar{y}_{FTRP})_{min} = \bar{y} \left(1 - \frac{1}{N}\right) \left[C_r^2 + V^2C_x^2 - 2pC_xC_r\right] \quad \ldots(3.4)$$

The symbols have their usual meaning and some specific symbols are already defined in Remark 1 and the proofs of above expressions are done in Appendix A as well.

**Remark 2:** For getting the optimum value of $k$ by solving the relation $\theta = V$, the cubic equation in the form of $k$ is given by:

$$(V - 1)k^3 + (f^2 + 20V + 32(\theta_1 - \theta_2)^2 - (10f + 12V + 6k)k + (8f^3 - 26V + 22) = 0$$

**Remark 3:** The relation $\theta_1 - \theta_2 = -\frac{V}{2f - 1}$ could be written as $f = \frac{1}{2}\left(1 - \frac{V}{\theta_1 - \theta_2}\right)$ \ldots(3.5)

Obviously, for a fix value of $k$ and $V$ we can calculate optimum value of sampling fraction $f$ and using this value one could get the optimum sample size for known population size $N$ since $f = \frac{n}{N}$. For example, at $k = 1, \theta_1 = 0, \theta_2 = 1$ and we have $f = \frac{1}{2}(1 + V) \Rightarrow n = \frac{N(1 + V)}{2}$.

4. Comparisons

(1) The variance of $\bar{y}$ in SRSWOR is given by

$$V(\bar{y}) = \left(\frac{1}{n} - \frac{1}{N}\right)S_r^2 = \bar{y} \left(\frac{1}{n} - \frac{1}{N}\right)C_r^2$$

and the minimum mean squared error (m.s.e.) of $\bar{y}_{FTRP}$ is:

$$M(\bar{y}_{FTRP})_{min} = \bar{y} \left(\frac{1}{n} - \frac{1}{N}\right) \left[C_r^2 + V^2C_x^2 - 2pC_xC_r\right]$$

Now, let $D = V(\bar{y}) - M(\bar{y}_{FTRP})_{min}$

$$D = \bar{y} \left(\frac{1}{n} - \frac{1}{N}\right)C_r^2 - \bar{y} \left(\frac{1}{n} - \frac{1}{N}\right) \left[C_r^2 + V^2C_x^2 - 2pC_xC_r\right]$$
A Study of Linear Combination Based Factor-Type Estimator for…

\[ -T\left(\frac{1}{n} - \frac{1}{N}\right)\left[\mathbf{Y}^{\top}\mathbf{C}_x^2 - 2\rho\mathbf{C}_x\mathbf{C}_r\right] \]

\( \bar{y}_{rnp} \) is better than \( \bar{y} \) if \( D_1 > 0 \)

\[ \Rightarrow -T\left(\frac{1}{n} - \frac{1}{N}\right)\left[\mathbf{Y}^{\top}\mathbf{C}_x^2 - 2\rho\mathbf{C}_x\mathbf{C}_r\right] > 0 \quad \Rightarrow \rho < 0 \]

Obviously, if there is negative correlation between \( X \) and \( Y \) then \( \bar{y}_{rnp} \) is better than \( \bar{y} \).

(2) The mean squared error of \( \bar{y}_{rnp} \) is:

\[ M(\bar{y}_{rnp}) = T\left(\frac{1}{n} - \frac{1}{N}\right)\left[\mathbf{C}_x^2 + \mathbf{C}_r^2 - 2\rho\mathbf{C}_x\mathbf{C}_r\right] \]

Let \( D_2 = M(\bar{y}_{rnp}) - M(\bar{y}_{rnp})_{\text{min}} \)

\[ = T\left(\frac{1}{n} - \frac{1}{N}\right)\left[\mathbf{C}_x^2 + \mathbf{C}_r^2 + 2\rho\mathbf{C}_x\mathbf{C}_r\right] - T\left(\frac{1}{n} - \frac{1}{N}\right)\left[\mathbf{C}_x^2 + \mathbf{C}_r^2 - 2\rho\mathbf{C}_x\mathbf{C}_r\right] \]

\( \bar{y}_{rnp} \) is better than \( \bar{y}_r \) if \( D_2 > 0 \quad \Rightarrow \rho > \frac{C_x}{C_r} \)

(3) Let \( D_3 = M(\bar{y}_r) - M(\bar{y}_{rnp})_{\text{min}} \)

\[ = T\left(\frac{1}{n} - \frac{1}{N}\right)\left[\mathbf{C}_x^2 + \left(\frac{n}{N-n}\right)\mathbf{C}_r^2 - 2\left(\frac{n}{N-n}\right)\rho\mathbf{C}_x\mathbf{C}_r\right] - T\left(\frac{1}{n} - \frac{1}{N}\right)\left[\mathbf{C}_x^2 + \mathbf{C}_r^2 - 2\rho\mathbf{C}_x\mathbf{C}_r\right] \]

\( \bar{y}_{rnp} \) is better than \( \bar{y}_r \) if \( D_3 > 0 \quad \Rightarrow \rho > \frac{C_x}{C_r} \)

(4) The mean squared error of \( \bar{y}_{sT} \) is:

\[ M(\bar{y}_{sT}) = T\left(\frac{1}{n} - \frac{1}{N}\right)\left[\mathbf{C}_x^2 + \left(\frac{n}{N-n}\right)\mathbf{C}_r^2 - 2\left(\frac{n}{N-n}\right)\rho\mathbf{C}_x\mathbf{C}_r\right] \]

Let \( D_4 = M(\bar{y}_{sT}) - M(\bar{y}_{rnp})_{\text{min}} \)

\[ = T\left(\frac{1}{n} - \frac{1}{N}\right)\left[\mathbf{C}_x^2 + \left(\frac{n}{N-n}\right)^2\mathbf{C}_r^2 - 2\left(\frac{n}{N-n}\right)^2\rho\mathbf{C}_x\mathbf{C}_r\right] - T\left(\frac{1}{n} - \frac{1}{N}\right)\left[\mathbf{C}_x^2 + \mathbf{C}_r^2 - 2\rho\mathbf{C}_x\mathbf{C}_r\right] \]

\( \bar{y}_{rnp} \) is better than \( \bar{y}_{sT} \) if \( D_4 > 0 \)

\[ \Rightarrow \rho^2 > \frac{2n\mathbf{C}_x\mathbf{C}_r}{N-n} \quad \Rightarrow \left[\mathbf{C}_x^2 + \mathbf{C}_r^2 - 2\left(\frac{n}{N-n}\right)^2\rho\mathbf{C}_x\mathbf{C}_r\right] > 0 \quad \Rightarrow \rho > \frac{C_x}{C_r} \quad \text{(4.1)} \]
which is a quadratic inequality in \( \rho \) and it will have real roots if

\[
\left( \frac{2nC_x^2C_{fi}^2}{N - n} \right) - 4C_x^2 \left( \frac{C_{fi}^2}{n - N} \right) - 1 > 0
\]

\[
\Rightarrow N(C_x^2 - C_{fi}^2)(2n - N) > 0 \quad \Rightarrow C_x > C_{fi} \quad \text{or} \quad n \left( N - n \right) > 0
\]

and the roots of \( \rho \) could be obtained by considering the inequality (4.1) as equality using the relation as below

\[
\rho = \frac{nC_x \pm \sqrt{N(C_x^2 - C_{fi}^2)(2n - N)}}{(N - n)C_x}
\]

(5) The optimum m.s.e. of factor-type estimator is

\[
M(\bar{Y}_{\text{ftp}})_{\text{ms}} = \left( \frac{1 - f}{n} \right) (1 - \rho^2) \sigma_b^2
\]

Again, let \( D_1 = M(\bar{Y}_{\text{ftp}})_{\text{ms}} - M(\bar{Y}_{\text{ftp}})_{\text{ms}} = 0 \) therefore, both the estimators are equal efficient.

5. Empirical Study

To examine the performance of the estimator \( \bar{Y}_{\text{ftp}} \) in comparison to the estimators \( \bar{Y}, \bar{Y}_p, \bar{Y}_r, \bar{Y}_{M}, \bar{Y}_{ry} \) and we have considered five data sets as A, B, C, D and E as given in the table-1:

<table>
<thead>
<tr>
<th>Table-1: Different Population under Consideration</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f )</td>
</tr>
<tr>
<td>( \bar{Y} )</td>
</tr>
<tr>
<td>( \bar{Y} )</td>
</tr>
<tr>
<td>( C_x )</td>
</tr>
<tr>
<td>( C_{fi} )</td>
</tr>
<tr>
<td>( \rho )</td>
</tr>
<tr>
<td>( N )</td>
</tr>
<tr>
<td>( n )</td>
</tr>
</tbody>
</table>
### Table: 2 Bias and M.S.E. of Different Estimators on Different Population

<table>
<thead>
<tr>
<th>Populations</th>
<th>Estimators</th>
<th>$k$</th>
<th>Bias</th>
<th>M.S.E.</th>
<th>PRE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population A</td>
<td>$\bar{y}$</td>
<td>---</td>
<td>0</td>
<td>167.4</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>$\bar{y}_x$</td>
<td>---</td>
<td>-1.7728</td>
<td>73.84</td>
<td>44.099</td>
</tr>
<tr>
<td></td>
<td>$\bar{y}_r$</td>
<td>---</td>
<td>4.3169</td>
<td>339.2</td>
<td>202.60</td>
</tr>
<tr>
<td></td>
<td>$\bar{y}_{st}$</td>
<td>---</td>
<td>-1.0039</td>
<td>166.3</td>
<td>99.339</td>
</tr>
<tr>
<td></td>
<td>$\bar{y}_{ftr}$</td>
<td>$k_i=2.3725$</td>
<td>-3.7401</td>
<td>55.23</td>
<td>32.987</td>
</tr>
<tr>
<td></td>
<td>$\bar{y}_{ftr}$</td>
<td>$k_i$</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td>$\bar{y}_{ftr}$</td>
<td>$k_i=1.2598$</td>
<td>-48.393</td>
<td>167.39</td>
<td>100.092</td>
</tr>
<tr>
<td></td>
<td>$\bar{y}_{ftr}$</td>
<td>$k_i$</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
</tbody>
</table>

| Population A |  $\bar{y}$   | --- | 0     | 8.9582 | 5.3500 |
|              |  $\bar{y}_x$ | --- | 0.0640 | 2.895 | 1.7290 |
|              |  $\bar{y}_r$ | --- | 0.2067 | 60.84 | 36.335 |
|              |  $\bar{y}_{st}$ | --- | -0.0229 | 9.076 | 5.4205 |
|              |  $\bar{y}_{ftr}$ | $k_i=1.5191$ | 0.0123 | 2.252 | 1.3451 |
|              |  $\bar{y}_{ftr}$ | $k_i=9.0124$ | 0.0030 | 2.252 | 1.3451 |
|              |  $\bar{y}_{ftr}$ | $k_i=2.4423$ | 0.1606 | 2.252 | 1.3451 |
|              |  $\bar{y}_{ftr}$ | $k_i=0.0032$ | 0.2512 | 8.970 | 5.3574 |
|              |  $\bar{y}_{ftr}$ | $k_i=1.3401$ | 0.1312 | 8.963 | 5.3533 |
|              |  $\bar{y}_{ftr}$ | $k_i$ | --- | --- | --- |

| Population C |  $\bar{y}$   | --- | 0     | 0.598 | 0.3574 |
|              |  $\bar{y}_x$ | --- | 0.0490 | 2.045 | 1.2213 |
|              |  $\bar{y}_r$ | --- | -0.0249 | 0.092 | 0.0553 |
|              |  $\bar{y}_{st}$ | --- | 0.0166 | 0.830 | 0.4958 |
|              |  $\bar{y}_{ftr}$ | $k_i=1.9551$ | -0.0363 | 0.132 | 0.0790 |
|              |  $\bar{y}_{ftr}$ | $k_i=1.9551$ | -0.0363 | 0.132 | 0.0790 |
|              |  $\bar{y}_{ftr}$ | $k_i$ | --- | --- | --- |
|              |  $\bar{y}_{ftr}$ | $k_i=10.9249$ | -1.6292E-05 | 0.5985 | 0.3574 |
|              |  $\bar{y}_{ftr}$ | $k_i$ | --- | --- | --- |

| Population D |  $\bar{y}$   | --- | 0     | 0.4035 | 0.2410 |
|              |  $\bar{y}_x$ | --- | 0.0255 | 0.7175 | 0.4285 |
|              |  $\bar{y}_r$ | --- | -0.0156 | 0.2408 | 0.1438 |
|              |  $\bar{y}_{st}$ | --- | 0.0039 | 0.4159 | 0.2484 |
|              |  $\bar{y}_{ftr}$ | $k_i=1.9700$ | -0.0115 | 0.2761 | 0.1649 |
|              |  $\bar{y}_{ftr}$ | $k_i$ | --- | --- | --- |
|              |  $\bar{y}_{ftr}$ | $k_i=12.3769$ | 0.0073 | 0.4034 | 0.2409 |
|              |  $\bar{y}_{ftr}$ | $k_i$ | --- | --- | --- |
5. Discussion and Conclusions

The Linear combination of Factor-Type estimator may be considered as an estimator of the population mean and properties like bias, m.s.e., etc. of the estimators could be derived in terms of population parameters under the concept of large sample approximations and a comparative study with existing estimators can be established. For the same we considered the linear combination of Factor-Type estimator and derive its bias and mean squared error. It is found that the proposed estimator is better over existing estimator if there is negative correlation between study and ancillary variable. i.e. suggested estimator have minimum mean squared error (m.s.e) in case the study and auxiliary variable are negatively correlated or equal efficient if not. Factor-type and Factor-type ratio product estimator both are equal efficient in term of parameters at optimum values of constant. Empirical study has been done over five populations and the bias and m.s.e. of suggested and existing estimators calculated. Percentage relative efficiency of the estimators has been calculated as well. We observed that where positive correlation between the variables then the suggested estimator is approximate equal efficient and for negative correlation it is better over existing. Also, there is the choice of $k$ for minimum bias and for specified value of $k$ we can obtain optimum choice of sample size.

Acknowledgement

Authors are thankful to the Editorial Board of JRSS and referees for recommending the manuscript for publication and their valuable suggestions.

References

APPENDIX A

Theorem 1: The form of the estimator is given as
\[
\hat{\theta} = f(k) + (1 - f)\beta(k)
\]
\[
\hat{\theta} = f\left[\frac{(k + C)}{(k + \beta)}\frac{\theta}{\hat{\theta}} + \frac{\theta}{\hat{\theta}}\right] + (1 - f)\beta\left[\frac{(k + C)}{(k + \beta)}\frac{\theta}{\hat{\theta}} + \frac{\theta}{\hat{\theta}}\right]
\]
The large sample approximation form of the above estimator is
\[
\hat{\theta} = (1 + e_f (f(1 + \theta_1) + \theta_2 e_f) + (1 - f)(1 + \theta_1 e_f) + \theta_2 e_f)^{-1}
\]
On solving it we get,
\[
\hat{\theta} = (1 + e_f (f(1 + \theta_1) + \theta_2 e_f) + (1 - f)(1 + \theta_1 e_f) + \theta_2 e_f)^{-1}
\]
ignoring higher order terms we get,
\[
\hat{\theta}_{FTRP} = \left[1 + e_f (\theta_1 - \theta_1) (2f - 1) e_f + (2f - 1) \rho C e_f \right]
\]

Theorem 2: The estimator \(\hat{\theta}_{FTRP}\) is biased and it is given by:
\[
\hat{\theta} = \hat{\theta} - \theta_1 \left[1 + \frac{1}{N}\left(\theta_1 - \theta_1\right) \left(2f - 1\right) e_f + \left(2f - 1\right) \rho C e_f \right]
\]

Proof: Since we know that
\[
\hat{\theta} = E(\hat{\theta} - \theta_1) \left[1 + \frac{1}{N}\left(\theta_1 - \theta_1\right) \left(2f - 1\right) e_f + \left(2f - 1\right) \rho C e_f \right]
\]

Theorem 3: The mean squared error of \(\hat{\theta}_{FTRP}\) is given by
\[
M(\hat{\theta}_{FTRP}) = \left(1 + \frac{1}{N}\left(\theta_1 - \theta_1\right) \left(2f - 1\right) e_f + \left(2f - 1\right) \rho C e_f \right)
\]

Proof: Since we know that
\[
M(\hat{\theta}_{FTRP}) = E\left(\hat{\theta}_{FTRP} - \theta_1\right)^2
\]

Theorem 4: The minimum M.S.E. of \(\hat{\theta}_{FTRP}\) at \(\theta_1 = \theta_2 = -\frac{V}{2f - 1}\) is
\[
M(\hat{\theta}_{FTRP})_{min} = \left(1 + \frac{1}{N}\left(\theta_1 - \theta_1\right) \left(2f - 1\right) e_f + \left(2f - 1\right) \rho C e_f \right)
\]

Proof: Diff. (3.3) w. r. to \(P = \theta_1 - \theta_1\) (say) and equate to zero.
\[
\frac{d}{dP} M(\hat{\theta}_{FTRP}) = \left(2f - 1\right) \rho C e_f - 2f - 1 = 0
\]
\[
\Rightarrow (\theta_1 - \theta_1) \left(2f - 1\right) e_f + \rho C e_f = 0
\]
\[
\Rightarrow (\theta_1 - \theta_1) = -\rho e_f \left(\frac{1}{2f - 1}\right) e_f = -\frac{V}{(2f - 1)}
\]

Therefore the minimum m.s.e. for \(\hat{\theta}_{FTRP}\) is given by
\[
M(\hat{\theta}_{FTRP})_{min} = \left(1 + \frac{1}{N}\left(\theta_1 - \theta_1\right) \left(2f - 1\right) e_f + \left(2f - 1\right) \rho C e_f \right)
\]