Journal of Reliability and Statistical Studies; ISSN (Print): 0974-8024, (Online):2229-5666
Vol. 5, Issue Special (2012): 51-61

SOME ASPECTS REGARDING TECHNIQUES OF STOCHASTIC APPROXIMATION IN SYSTEMS ANALYSIS. A SURVEY

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Dedicated to B.V. Gnedenko on the occasion of 100 years since his birth

Abstract

As it is known the stochastic-approximation procedures require very little prior knowledge of the process and achieve reasonably good results. For this reason such methods work satisfactorily in various applications. Many and very important results are obtained, in particular, by Gnedenko, Solovyev, Venter and Gastwirth. In this paper we refer to some aspects regarding to the problem of the increase of the effectiveness of stand-by systems as a way in which the stochastic-approximation techniques can be applied in practice. Also some aspects concerning the stochastic-approximation plan will be discussed.

Key words: Control processes, Diffusion processes, Stochastic approximation processes, Stochastic differential equations, Stand-by systems.

1. Introduction

As it is well known, in the last decades the stochastic approximation procedures have been developed very rapidly and have found many and various applications. Generally speaking, it can be considered a problem where computation is split among several processors, operating and transmitting data to one another asynchronously. Such algorithms are only being to come into prominence, due to both the developments of decentralized processing and applications where each of several locations might control or adjusted *local variable* but the criterion of concern is global.

At the same time, in the present-day technology, reliability of equipment is increased by employing the method of *stand-by systems* that is the introduction of extra components, units and entire assemblies. Thus, the purpose of the supplementary devices is to take over operation if the basic systems break down.

Depending on the state of the stand-by equipment, we distinguish loaded, nonloaded and partially loaded relief. In the case of loaded relief, the stand-by unit is in the same state as the operating unit and for this reason has the same intensity of breakdowns. In the partially loaded case, the stand-by device is loaded, but not so fully as the main equipment and for this reason has a different breakdown intensity. A stand-by unit that is not loaded does not, naturally, suffer breakdown. In this sense, many and very important results have been obtained especially by B. V. Gnedenko and A. D. Solovyev.

Some aspects of these studies will be discussed, briefly, in Section 3.

On the other hand, let us consider a *system* or *item*, with a *life time* that has the distribution function $F(t)$, and which is inspected at times $t_1, t_2, \cdots$. Now, if inspection reveals that the system is inoperative, it is repaired or replaced. Nothing is done otherwise. So, a general problem is to choose the inspection plan, that is to say the sequence $t_1, t_2, \cdots$ in an optimal way in a suitable sense.

A criterion of optimality is defined by J.H. Venter and J.L. Gastwirth who have proved that a stochastic-approximation plan satisfies the criterion. The subject is also developed by M.T. Wasan.

In the last section we shall refer to this problem by considering the condition *vi* of Section 3.1.

2. A Short Review on Stochastic Differential Equations

To describe the motion of a particle driven by a *white noise* type of force (due to the collision with the smaller molecules of the fluid) the following equation is used

$$\frac{d\mathbf{v}(t)}{dt} = -\beta\mathbf{v}(t) + \mathbf{f}(t) \qquad (1)$$

where v is the particle's velocity (for a spherical particle of radius a), $\beta = 6\pi a\eta / m$ (η is the coefficient of dynamical viscosity of the surrounding fluid, and m is the particle's mass) and $\mathbf{f}(t)$ is the white noise term.

Equation (1) is referred to as the *Langevin's equation*. Its solution is the following

$$\mathbf{v}(t) = \mathbf{v}_0 e^{-\beta t} + \int_0^t e^{-\beta(t-s)}\mathbf{f}(s)ds. \qquad (2)$$

If we denote by $\mathbf{w}(t)$ the Brownian motion, then it is given by

$$\mathbf{w}(t) = \frac{1}{q}\int_0^t \mathbf{f}(s)ds, \qquad (3)$$

so that $\mathbf{f}(s) = \frac{qd\mathbf{w}(s)}{ds}$. But $\mathbf{w}(t)$ is nowhere differentiable, so that $\mathbf{f}(s)$ is not a function. Therefore, the solution (2) of Langevin's equation is not a well-defined function. This difficulty can be overcome, in the simple case, as follows. Integrating (2) by parts, and using (3), it results

$$\mathbf{v}(t) = \mathbf{v}_0 e^{-\beta t} + q\mathbf{w}(t) - \beta q\int_0^t e^{-\beta(t-s)}\mathbf{w}(s)ds. \qquad (4)$$

All functions in (4) are well defined and continuous, so that the solution (3) can be interpreted as having the meaning of (4). Now, such a procedure can be generalized in the following way. Let us consider two functions $f(t)$ and $g(t)$ that are defined for $a \le t \le b$. For any partition

$$P : a \le t_0 < t_1 < \cdots < t_n$$

we define

$$S_P = \sum_{i=1}^{n} f(\xi_i)[g(t_i) - g(t_{i-1})]$$

where $t_{i-1} \le \xi_i \le t_i$.
If a limit

$$\lim_{|P| \to 0} S_P = I$$

exists, where

$$|P| = \max_{1 \le i \le n}(t_i - t_{i-1})$$

then, it is said that I is the *Stieltjes integral* of $f(t)$ with respect to $g(t)$ and it is denoted by

$$I = \int_a^b f(t) dg(t).$$

Now the stochastic differential equation

$$\begin{aligned} dx(t) &= a(x(t),t)dt + b(x(t),t)dw(t) \\ x(0) &= x_0 \end{aligned} \tag{5}$$

is defined by the Itô integral equation

$$x(t) = x_0 + \int_0^t a(x(s),s)ds + \int_0^t b(x(s),s)dw(s). \tag{6}$$

The simplest example of a stochastic differential equation is the following equation

$$\begin{aligned} dx(t) &= a(t)dt + b(t)dw(t) \\ x(0) &= x_0 \end{aligned} \tag{7}$$

which has the solution

$$x(t) = x_0 + \int_0^t a(s)ds + \int_0^t b(s)dw(s).$$

The *transition probability density* of $x(t)$ is a function $p(x,s;y,t)$ satisfying the condition

$$P(x(t) \in A | x(s) = x) = \int_A p(x,s;y,t)dy$$

for $t > s$ where A is any set in R. It is supposed that $a(t)$ and $b(t)$ are deterministic functions.
The stochastic integral

$$\chi(t) = \int_0^t b(s)dw(s)$$

is a limit of linear combinations of independent normal variables

$$\sum_i b(t_i)[w(t_{i+1}) - w(t_i)].$$

Thus, the integral is also a normal variable.
But, then

$$\chi(t) = x(t) - x_0 - \int_0^t a(s)ds$$

is a normal variable, and therefore

$$p(x,s;y,t) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(y-m)^2}{2\sigma}}$$

with

$$m = E(x(t)|x(s) = x).$$

Now

$$E(x(t)|x(s) = x) = x + \int_s^t a(u)du$$

is the expectation of the stochastic integral vanishes. As for the variance it is given by the relation

$$\sigma = Var\, x(t) \;= E\left[\int_s^t b(u)dw(u)\right]^2 = \int_s^t b^2(u)du.$$

In conclusion, $p(x,s;y,t)$ is given by the following equation

$$p(x,s;y,t) = \left[2\pi \int_s^t b^2(u)du\right]^{-1/2} \cdot e^{-\frac{\left(y-x-\int_s^t a(u)du\right)^2}{2\int_s^t b^2(u)du}}.$$

For more details and proofs see, for example: Da Prato G. and Zabczyk J. (1992), Da Prato G. (1998), Ikeda N. and Watanabe S. (1989), Itô K. and McKean H. P. Jr. (1996), Øksendal B. (2003), Schuss Z. (1980).

3. Considerations on the increase of the effectiveness of stand-by systems

As we have already emphasized, some problems and results concerning the increase of the effectiveness of stand-by systems, due especially to B. V. Gnedenko and A. D. Solovyev are discussed, in short, below.

There are enough situations when it is possible to have an entire device in reserve as, for example, a generator at a power station. Also it is possible to have in reserve a component of a system or even a single element. A question arises: *what is* *preferable, to have large units or single elements in reserve?* An answer is given in the following theorem:

Theorem 1. *If the switching of stand-by devices (units, elements, a.s.o.) is flawless, then both in the case of loaded and nonloaded relief, an increase in the scale of the stand-by system reduces non-breakdown operation of the whole system.*

3.1. The Probability that the System will Operate Flawlessly

To increase the effectiveness of stand-by systems, devices that have failed are repaired. Hence it is very interesting to investigate the effect of repair on increasing the reliability. We confine ourselves to the case of one basic and one reserve system.

Let us assume that the following conditions are fulfilled:

i. on breakdown of the basic device, the stand-by unit immediately takes up the load;
ii. the device that has failed undergoes repair immediately;
iii. the repairs fully restore the properties of the basic device that failed;
iv. the repair time is a random variable with a distribution function $G(x)$;
v. the repaired device becomes a stand-by unit;
vi. the period of faultless operation of the device is random and is distributed in accord with the law $F(x) = 1 - e^{-\lambda x}$, $\lambda > 0$, for the basic device and in accord with the law $F_1(x) = 1 - e^{-\lambda_1 x}$, $\lambda_1 \geq 0$, for the stand-by device. In particular, if the stand-by unit is nonloaded then, $\lambda_1 = 0$ and if it is loaded then $\lambda_1 = \lambda$.

We retain the following definition:

Definition 1. *It is said that the system (basic unit plus stand-by unit) breaks down if both devices go out of commission at the same time.*

Now let us denote by *P(x)* the probability that the system will operate flawlessly for a time greater than *x*. Also the Laplace transforms are introduced

$$g(s) = \int_0^\infty e^{-sx} dG(x), \qquad \varphi(s) = -\int_0^\infty e^{-sx} dP(x)$$

Regarding the probability *P(x)* the following result is obtained:

Theorem 2. *Under the conditions i-vi, the probability P(x) satisfies the integral equation*

$$P(x) = e^{-(\lambda+\lambda_1)x} + (\lambda+\lambda_1)e^{-\lambda x}\int_0^x e^{-\lambda_1 z}[1 - G(x-z)]dz + \\ + (\lambda+\lambda_1)\int_0^x \int_0^{x-y} e^{-(\lambda+\lambda_1)y-\lambda z} P(x-y-z)dG(z)dy. \tag{8}$$

Let us observe that the solution of the equation (8) is the following:

Lemma 1. *In terms of Laplace transforms, the solution of the equation (8) is given by the formula*

$$\varphi(s) = \frac{\lambda(\lambda+\lambda_1)[1-g(\lambda+s)]}{(\lambda+s)[s+(\lambda+\lambda_1)(1-g(\lambda+s))]}. \tag{9}$$

By virtue of the properties of the exponential distribution, the result can be extended to the case when there are *n* operating devices and one stand-by unit. All devices have the same properties, namely they have the same distribution functions for operating time and repairs. To see this it is necessary only to replace λ by $n\lambda$ in (8) and (9).

On the other hand, to calculate the expectation of the time of flawless operation of the system, consider $\left[\frac{d\varphi(s)}{ds}\right]_{s=0}$ which is given by the equality

$$\left[\frac{d\varphi(s)}{ds}\right]_{s=0} = -\frac{[\lambda(\lambda+\lambda_1)(1-g(\lambda))][\lambda+(\lambda+\lambda_1)(1-g(\lambda))]}{[\lambda(\lambda+\lambda_1)(1-g(\lambda))]^2}.$$

Now

$$m = -\left[\frac{d\varphi(s)}{ds}\right]_{s=0} = \frac{\lambda+(\lambda+\lambda_1)(1-g(\lambda))}{\lambda(\lambda+\lambda_1)(1-g(\lambda))} \tag{10}$$

and we come to the following conclusion: for a nonloaded stand-by system we have $\lambda_1 = 0$, so that

$$m_1 = \frac{\lambda+\lambda(1-g(\lambda))}{\lambda^2(1-g(\lambda))} = \frac{2-g(\lambda)}{\lambda(1-g(\lambda))}, \tag{11}$$

while for a loaded stand-by system $\lambda_1 = \lambda$ and one gets

$$m_2 = \frac{\lambda+2\lambda(1-g(\lambda))}{2\lambda^2(1-g(\lambda))} = \frac{3-2g(\lambda)}{2\lambda(1-g(\lambda))}. \tag{12}$$

3.2. Two Main Results

In most practical cases, the mean duration of repairs is considerably less than the mean time of flawless operation of the device. For this reason it was observed that some limit theorems are necessary just to give a precise and rigorous meaning to the results obtained in these situations. We shall refer, in brief, to such situations.

Let us suppose that the function $G(x)$ depends on a certain parameter ν and for any $\varepsilon > 0$,

$$1 - G_\nu(\varepsilon) \to 0 \quad \text{as} \quad \nu \to \infty. \tag{13}$$

The converse is also true because, if for any $s > 0$ we have the relation $g_\nu(s) \to 1$ as $\nu \to \infty$ then, for any $x > 0$,

$$G_\nu(x) \to 1$$

as $\nu \to \infty$.

Now let us define

$$\alpha_\nu = \left(1 + \frac{\lambda_1}{\lambda}\right)(1 - g_\nu(\lambda))$$

from which we get

$$\lambda + \lambda_1 = \frac{\lambda \alpha_\nu}{1 - g_\nu(\lambda)}. \tag{14}$$

By (9) we have

$$\varphi_\nu(\alpha_\nu s) = \frac{\lambda(\lambda + \lambda_1)[1 - g_\nu(\lambda + \alpha_\nu s)]}{(\lambda + \alpha_\nu s)[\alpha_\nu s + (\lambda + \lambda_1)(1 - g_\nu(\lambda + \alpha_\nu s))]}$$

and replacing $\lambda + \lambda_1$ from (14) it follows that

$$\varphi_\nu(\alpha_\nu s) = \frac{\lambda^2 \dfrac{1 - g_\nu(\lambda + \alpha_\nu s)}{1 - g_\nu(\lambda)}}{(\lambda + \alpha_\nu s)\left(s + \lambda \dfrac{1 - g_\nu(\lambda + \alpha_\nu s)}{1 - g_\nu(\lambda)}\right)}. \tag{15}$$

Thus the following theorem is obtained:

Theorem 3. *If conditions (8), (9) and (10) to (15) hold then, by the condition (13), the flow of failures of a reduplicated system tends to the elementary case, given the choice of a proper unit of time.*

Also the effect of repair on the operational effectiveness of a system can be estimated. To this end it is natural to consider the ratio of the mean operational time of a system with repair to that without repair. From the formula (10) the former can be calculated and from the formula

$$a_0 = \frac{2\lambda + \lambda_1}{\lambda(\lambda + \lambda_1)}$$

the latter. The effectiveness of repair is now given by the equality

$$e_\nu = \frac{\lambda + (\lambda + \lambda_1)(1 - g_\nu(\lambda))}{\lambda(\lambda + \lambda_1)(1 - g_\nu(\lambda))} \cdot \frac{\lambda(\lambda + \lambda_1)}{2\lambda + \lambda_1} = \frac{\lambda + (\lambda + \lambda_1)(1 - g_\nu(\lambda))}{(2\lambda + \lambda_1)(1 - g_\nu(\lambda))}. \tag{16}$$

Suppose now that $m_1(\nu) = \int_0^\infty x dG_\nu(x) = 1/\nu$, $\quad m_2(\nu) = \int_0^\infty x^2 dG_\nu(x) < +\infty$ and

$$\frac{m_2(\nu)}{m_1(\nu)} \to 0 \tag{17}$$

as $\nu \to \infty$. Then, the following theorem holds:

Theorem 4. *Let us suppose that conditions (8), (9), (10) to (17) are satisfied. Then for* ν *sufficiently large, the mean time of flawless operation of a system with stand-by relief is asymptotically equal to the mean time of the system under the assumption that*

$$G_{\nu}(x) = 1 - e^{-\nu x}.$$

For more details and proofs see Gnedenko (1964), (1976); Solovyev (1964).

4. A Problem of Stochastic Approximation

The basic Stochastic Approximation Algorithms introduced by H. Robbins & S. Monro and by J. Kiefer & J. Wolfowitz have been the subject of an enormous literature.

This is due to the large number of applications and the interesting theoretical issues in the analysis of d*ynamically defined stochastic processes*.

In recent years, algorithms of the stochastic approximation type have found applications in new and diverse areas, and new techniques have been developed for proofs of convergence and rate of convergence. The actual and potential applications in signal processing have exploded. Indeed, whether or not they are called stochastic approximations, such algorithms occur frequently in practical systems for the purposes of noise or interface cancellation, the optimization of *post processing* or *equalization* filters in time varying communication channels, adaptive antenna systems, and many related applications.

In such applications, the underlying processes are often nonstationary, the optimal value of the parameter of the system (say, for example θ) changes with time, and we keep the *step size* (say for example ε_n) strictly away from zero in order to allow *tracking*. Such tracking applications lead to new problems in the asymptotic analysis (e.g. when ε_n are adjusted adaptively); one wishes to estimate the tracking errors and their dependence on the structure of the algorithm.

Now let us return to the condition *vi* in Section 3.1 and, for an unknown parameter $\lambda > 0$, let us consider a distribution function defined as follows

$$\begin{aligned} F(x) &= 1 - e^{-\lambda x} \quad \text{for} \quad x \geq 0 \\ F(x) &= 0 \quad \text{for} \quad x < 0. \end{aligned} \tag{18}$$

$F(x)$ can be the distribution function of a *system* or *item*, with a *life time* for which inspections are made at time $t_1, t_2, t_3, \cdots$. If the conclusion of the inspections is that the system is inoperative, then it will be repaired or replaced. In any other case nothing is done. Thus, the problem is to choose the inspection plan, namely to choose the sequence $t_1, t_2, t_3, \cdots$ in an optimal way in a suitable sense.

In this way a problem of stochastic approximation, initially studied by J.H. Venter and J.L. Gastwirth, and also discussed by M.T. Wasan, is obtained.

Some aspects of their studies are discussed, in short, below.

Let $0 < a < b$ two constants and let us suppose that $a < \lambda < b$. The inspection times are defined as follows

$$T_1 = t_1, \quad T_i = t_i - t_{i-1}, \quad i = 2,3,4,\cdots. \tag{19}$$

Now let us consider an arbitrary sequence of random variables $\{X_n\}$ for which the joint distribution of any finite number does not depend on λ . Denote $T_1 = \max\{0, X_1\}$ and one defines $\{T_n\}$ as follows

$$T_{n+1} = \max\{0, f_n(Y_1,\cdots,Y_n) + X_n\} \quad \text{for} \quad n = 1,2,3,\cdots \tag{20}$$

where f_n is a real-valued measurable function of $(Y_1,\cdots,Y_n)$, functionally independent of λ and Y_i, $i = 1,2,3,\cdots,$ are random variables with conditional distribution

$$Y_i : \begin{pmatrix} 0 & 1 \\ 1 - e^{-\lambda T_i} & e^{-\lambda T_i} \end{pmatrix} \tag{21}$$

given $\{Y_1,\cdots,Y_{i-1},T_1,\cdots,T_i\}$. Hence one has $Y_i = 1$ if the i[th] inspection conducts to the conclusion that the system is operative and $Y_i = 0$ if it is inoperative.

We can understand that after n inspections, the next inspection time T_{n+1} depends on the past observations $(Y_1,\cdots,Y_n)$ through f_n while X_n allows for additional randomization.

Let us denote by $\mathcal{G}$ the class of all inspection plans and let I be a generic element of $\mathcal{G}$. Now, a criterion of optimality can be obtained (following the plan of M.T. Wasan).

Let $J_n(I,\lambda)$ be defined by the equality

$$J_n(I,\lambda) = \frac{E\left(\dfrac{d}{d\lambda}\log L_n(\lambda)\right)^2}{n} \tag{22}$$

the average information obtainable from a plan I after n inspections, where $L_n(\lambda)$ is the probability function of λ based on $(Y_1,Y_2,\cdots,Y_n,T_1,T_2,\cdots,T_n)$. At the same time, the limiting average information $J(I,\lambda)$, obtainable from plan I, is defined by the following equality

$$J(I,\lambda) = \liminf_{n\to\infty} J_n(I,\lambda). \tag{23}$$

Now, the following question arises: *how one can maximizing* $J_n(I,\lambda)$ *and* $J(I,\lambda)$ *by a judicious choice of* I ? Hence we attain to a well known method of efficient estimation of λ , which conduct us to the theorem below:

Theorem 5.

$$J_n(I,\lambda) \le \frac{T_\lambda(2-\lambda T_\lambda)}{\lambda} \qquad (24)$$

for each n and for all λ *and I, where* T_λ *is the solution of the equation*

$$\lambda T = 2(1-e^{-\lambda T}). \qquad (25)$$

[$T_\lambda = -\log p/\lambda$ *and* T_λ *is the 100(1-p)*th *percentile of the exponential distribution where* $p \approx 0.203$ *.]*

Let us observe that the equality in (24) follows if and only if $T_i = T_\lambda$ with probability one for each *i*. That is to say if λ were known, the optional inspection plan in the sense of maximizing $J_n(I,\lambda)$ for each *n* and λ would call for periodic inspection with interinspection times T_λ . But within the class $\mathcal{G}$ (which means that λ is unknown) there is no optimal plan for which the equality is obtained in (24).

An optimality criterion can be defined by using the concept of *adaptive inspection plan*.

Definition 2*. An inspection plan I is said to be adaptive, relative to* $J(I,\lambda)$ *, if the following equality holds*

$$J(I,\lambda) = \frac{T_\lambda(2-\lambda T_\lambda)}{\lambda}. \qquad (26)$$

A stochastic-approximation plan which is adaptive, denoted by *SA* is defined and the following main result is obtained:

Theorem 6. *The SA plan is adaptive.*

For details and proofs see Wasan (1969); Venter and Gastwirth (1964).
Remark: We emphasized here only some aspects of the problem in which the stochastic-approximation techniques can be applied to solve some practical problems. Many other examples can be discussed in this way, such topics being of actuality in our days.

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