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ON SOME CHARACTERISTICS OF GEOMETRIC PROCESSES

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Abstract

In this paper a method for reliability analysis of restorable items is considered. We present the model describing a variation of reliability characteristics of objects and taking into account incomplete repair of operability after failure. The asymptotic solution for the intensity of the geometric process model is obtained. Reliability characteristics of the geometric process model for various distribution laws and various parameters are calculated.

Key words: Distribution law, Geometric process, Incomplete repair, Reliability, Statistical model.

1. Introduction

In the present report we consider a method for reliability analysis of restorable objects. Suppose that the restorable object is repaired after failure. During the repair its operability is recovered. It is important to note that the ageing can take place even at initial stages of the object functioning and leads to degradation of reliability characteristics. This in turn affects the repair and leads to incomplete repair of the object's operability.

Let us consider some generalized object structurally consisting of details, elements, subcells etc. Operability recovery is performed as follows. The failed element is replaced by a new one. It is important to note that operable details of the failed object have already spent a part of resource. Therefore we can assume that the restored object cannot have the same reliability characteristics as the ones before the functioning start-up. It is important to note that complete recovery of operability does not occur when a scheduled preventive maintenance is carried out.

We define the following concept. The repair is called *incomplete* if after a complex of regenerative actions, the objects' reliability characteristics are essentially above the ones at the moment of failure but lower than the reliability characteristics of a new element.

The outline of the paper is as follows. In the next section we describe the model of geometric process. In Section 3 we present the renewal equations for intensity and renewal function. Section 4 is devoted to the study of the limiting distribution of τ_∞. The correlation coefficients of the instances of the nth and mth failure times (τ_n & τ_m) are obtained and investigated. In Section 5 we consider the asymptotic behavior of the renewal function and intensity. In Section 6 we introduce and describe new characteristics, namely the geometric of renewal function (GRF) and the geometric of intensity function (GIF). Section 7 summarizes our findings.

2. Geometric Process

Let us consider the model describing the variation of object reliability characteristics and taking into account incomplete repair of operability after failure. The behavior of complex systems is well described by this model. We assume that the system consists of components, subsystems, system parts.

Consider the functioning process of the system. A failed element is found and is replaced by a new one. The system passes again in operating state. We should take into consideration that all elements and system parts (except the replaced element) had a certain operating time and have already spent a part of the resource. Thus we should take into account incomplete recovery considering the system functioning process as a whole.

Definition 1. *Let* $\{\xi_i\}, i \geq 1$ *be a sequence of independent random variables. Each* ξ_i *corresponds to operating time between failures of the object with the distribution function* $F_{\xi_i}(t)$ *generated by the distribution* $F(t)$ *as follows*

$$F_{\xi_i}(t) = F\left(\frac{t}{\gamma^{i-1}}\right), i = 1, 2, ..., \tag{1}$$

where γ *is a positive constant. Then, the sequence* $\{\xi_i\}, i \geq 1$ *is called geometric process.*

Further we denote by γ the common ratio of the geometric process (or the process ratio).

Geometric processes in a reliability context were introduced by Lam (1988). Saenko (1994) considered an application of the geometric process to the model of alternating renewal process. Finkelstein (1993) considered some generalizations by applying geometric processes to a nonlinear scale of transformation. Antonov et. al. (2007) described the maximum likelihood method for the estimation of parameters for the geometric process model. Estimations of the index γ were obtained for various distribution laws of the first operating time.

Consider the following functioning strategy of an object. The object operates during a random time. After failure the object is restored and we consider that the restoration is incomplete. The *degradation coefficient* γ characterizes the incompleteness of repair. We suppose that the repair time of the object is negligible in comparison with the operating time between failures, i.e. the repair time is immediate.

3. Renewal Equation

As a result of incomplete repair the operating time of the renewal object is reduced (by probability) by γ times in comparison with the previous operation phase:

$$\xi_2 \stackrel{d}{=} \gamma\xi_1, ..., \xi_n \stackrel{d}{=} \gamma^{n-1}\xi_1, 0 < \gamma \leq 1 .$$

Mathematical dependence between distribution functions of the operating time between failures of the restored object (taking into account incomplete repair) can be expressed as

$$F_{\xi_2}(t) = F_{\xi_1}\left(\frac{t}{\gamma}\right), ..., F_{\xi_n}(t) = F_{\xi_1}\left(\frac{t}{\gamma^{n-1}}\right),$$

where $F_{\xi_i}(t)$ is the distribution function of the operating time between failures of $(i-1)$ times restored object and γ is the coefficient of incomplete restoration (degradation coefficient). Then, distribution densities are related through the following equation

$$f_{\xi_n}(t)=\frac{1}{\gamma^{n-1}}f_{\xi_1}\left(\frac{t}{\gamma^{n-1}}\right).$$

The degradation coefficient γ is an average value that reflects the accumulated process of damages and defects and indirectly characterizes the process of gradual material weariness, physical ageing, wear ability, corrosion, etc.

Let us define the expression that establishes the relationship between the failure rate at the initial stage of operation and the failure rate after the $(n-1)^{\text{th}}$ failure. By definition the failure rate can be defined by

$$\lambda(t)=\frac{f(t)}{1-F(t)}.$$

Then the expression of the failure rate of $(n-1)$ times restored object can be written as

$$\lambda_{\xi_n}(t)=\frac{f_{\xi_n}(t)}{1-F_{\xi_n}(t)}=\frac{\frac{1}{\gamma^{n-1}}f_{\xi_1}\left(\frac{t}{\gamma^{n-1}}\right)}{1-F_{\xi_1}\left(\frac{t}{\gamma^{n-1}}\right)}=\frac{1}{\gamma^{n-1}}\lambda_{\xi_1}\left(\frac{t}{\gamma^{n-1}}\right).$$

Thus, after each restoration the failure rate becomes $1/\gamma$ times more than the failure rate during the previous time interval. The time scale of the process also changes. Such model allows the evaluation of the reliability characteristics for each operability interval such as the object survival function after the i^{th} restoration.

Note that the given model can be used to estimate object reliability characteristics at the stage of infant mortality when effects of a rejuvenation system are observed. At the stage of infant mortality the degradation coefficient has to be more than 1 unit. Then, the degradation coefficient represents the average of times each sequential operation time is greater than the preceding one.

In this paper, we consider calculation methods of reliability characteristics for restored objects, like the intensity and cumulative intensity functions.

The *Renewal (cumulative intensity) function* (RF) is defined by

$$\Omega(t)=\sum_{n=1}^{\infty}F_{\xi}^{(n)}(t)=\sum_{n=1}^{\infty}F_n(t), \tag{2}$$

where $F_{\xi}^{(n)}(t)=F_{\xi}^{(n-1)}(t)*F_{\xi_n}(t)$ is the n^{th} order convolution of the distribution functions of the operating times between failures, $F_n(t)=F_{\xi}^{(n)}(t)=Pr(\tau_n<t)$ is the cumulative distribution function of $\tau_n=\sum_{i=1}^{n}\xi_i$ which is the instant of the n^{th} failure ($F_{\xi}^{(0)}(t)=1$).

The *intensity function* (IF) is defined by

$$\omega(t)=\frac{d\Omega}{dt}=\sum_{n=1}^{\infty}f_{\xi}^{(n)}(t)=\sum_{n=1}^{\infty}f_n(t), \tag{3}$$

where $f_{\xi}^{(n)}(t)$ is the convolution of densities of the operating times between failures, $f_n(t)=f_{\xi}^{(n)}(t)=\frac{d}{dt}Pr(\tau_n<t)$ is the density function of $\tau_n=\sum_{i=1}^{n}\xi_i$.

Unfortunately for the given model analytical solutions for RF and IF cannot be obtained even for standard distributions like the exponential. The solution can be obtained only by numerical methods.

We find now the asymptotic solution for given measures. Recall that the $(n+1)^{\text{th}}$ and the n^{th} operating times between failures satisfy the condition $\xi_{n+1} \overset{d}{=} \gamma\xi_n$, where γ is a positive constant and $\xi_1, \xi_2, \ldots$ are independent random variables.

Perform some intermediate calculation and obtain

$$f_n(t) = \tfrac{d}{dx} F_n(t), \text{ n=2, 3,...}, \quad f_1(t) = \tfrac{d}{dx} F_1(t) = f(t),$$

$$F_n(t) = \int_0^t F_{n-1}\left(\frac{t-u}{\gamma}\right) f(u)\,du = \int_0^{t/\gamma} f_{n-1}(\tau) F(t-\gamma\tau)\,d\tau,$$

Finally we get the expression for the density convolution

$$f_n(t) = \frac{1}{\gamma}\int_0^t f_{n-1}\left(\frac{t-u}{\gamma}\right) f(u)\,du = \int_0^{t/\gamma} f_{n-1}(\tau) f(t-\gamma\tau)\,d\tau. \qquad (4)$$

Using (2) and (3) we obtain the formulas for IF and RF:

$$\Omega(t) = F(t) + \int_0^t \sum_{n=2}^{\infty} F_{n-1}\left(\frac{t-u}{\gamma}\right) f(u)\,du = F(t) + \int_0^t \Omega\left(\frac{t-u}{\gamma}\right) f(u)\,du =$$

$$= F(t) + \int_0^{t/\gamma} \sum_{n=2}^{\infty} f_{n-1}(\tau) F(t-\gamma\tau)\,d\tau = F(t) + \int_0^{t/\gamma} \omega(\tau) F(t-\gamma\tau)\,d\tau$$

and

$$\omega(t) = f(t) + \int_0^t \sum_{n=2}^{\infty} f_{n-1}\left(\frac{t-u}{\gamma}\right) f(u)\,d\frac{u}{\gamma} = f(t) + \frac{1}{\gamma}\int_0^t \omega\left(\frac{t-u}{\gamma}\right) f(u)\,du =$$

$$= f(t) + \int_0^{t/\gamma} \sum_{n=2}^{\infty} f_{n-1}(\tau) f(t-\gamma\tau)\,d\tau = f(t) + \int_0^{t/\gamma} \omega(\tau) f(t-\gamma\tau)\,d\tau. \qquad (5)$$

Find now the Laplace transformation for RF and IF using the expression $\overline{F}_{\xi_n}(p) = \gamma^{n-1}\overline{F}\left(\gamma^{n-1}p\right) = \frac{1}{p}\,\overline{f}\left(\gamma^{n-1}p\right)$. Then,

$$\overline{\Omega}(p) = \sum_1^{\infty} \overline{F}_n(p) = \frac{1}{p}\sum_{n=1}^{\infty}\prod_{k=1}^{n} \overline{f}_{\xi_k}(p) = \frac{1}{p}\sum_{n=1}^{\infty}\prod_{k=1}^{n} \overline{f}\left(\gamma^{k-1}p\right),$$

$$\overline{\omega}(p) = \sum_{n=1}^{\infty}\prod_{k=1}^{n} \overline{f}\left(\gamma^{k-1}p\right). \qquad (6)$$

Equations (6) can be inverted numerically (Braun et. al., 2005]. Note that the IF for $0 < \gamma < 1$ and for sufficiently large t can be non-finite. However it is always finite for $\gamma > 1$.

Applying the iteration method we find the particular solutions. For the first step put

$$f(t) = \lambda \exp(-\lambda t).$$

As initial approximation take $\omega_0(t) \equiv 0$ and write the expression for ω_1

$$\omega_1(t) = f(t) = \lambda \exp(-\lambda t).$$

From (4) find the new value of the density

$$f_2(t)=\int_0^{t/\gamma}\lambda e^{-\lambda\tau}\lambda e^{-\lambda(t-\gamma\tau)}d\tau=\lambda^2e^{-\lambda t}\int_0^{t/\gamma}e^{\lambda(t-\gamma\tau)}d\tau=\frac{\lambda}{\gamma-1}\left(e^{-\frac{\lambda t}{\gamma}}-e^{-\lambda t}ght\right).$$

Further, solving the integral equation (5) again we get the expression for IF

$$\omega_2(t)=f(t)+f_2(t)=\lambda e^{-\lambda t}+\frac{\lambda}{\gamma-1}\left(e^{-\frac{\lambda t}{\gamma}}-e^{-\lambda t}\right)=\frac{\lambda}{\gamma-1}e^{-\frac{\lambda t}{\gamma}}+\frac{\lambda(\gamma-2)}{\gamma-1}e^{-\lambda t}.$$

In such a manner we find new values of the distribution density for the operating time between failures and redefine the intensity

$$f_3(t)=\int_0^{t/\gamma}f_2(\tau)f(t-\gamma\tau)d\tau=\frac{\lambda^2e^{-\lambda t}}{\gamma-1}\int_0^{t/\gamma}\left(e^{\frac{\lambda(\gamma^2-1)}{\gamma}\tau}-e^{\lambda(\gamma-1)}\right)d\tau=$$

$$=\frac{\lambda\gamma}{(\gamma-1)^2}(\gamma+1)e^{-\frac{\lambda x}{\gamma^2}}+\frac{\lambda}{(\gamma-1)^2}(\gamma+1)e^{-\lambda x}-\frac{\lambda}{(\gamma-1)^2}e^{-\frac{\lambda x}{\gamma}};$$

$$\omega_3(t)=f(t)+f_2(t)+f_3(t)=\frac{\lambda}{\gamma-1}e^{-\frac{\lambda t}{\gamma}}+\frac{\lambda(\gamma-2)}{\gamma-1}e^{-\lambda t}+f_3(t)=$$

$$=\frac{\lambda\gamma}{(\gamma-1)^2}(\gamma+1)e^{-\frac{\lambda t}{\gamma^2}}+\frac{\lambda(\gamma-2)}{(\gamma-1)^2}e^{-\frac{\lambda t}{\gamma}}+\frac{\lambda(\gamma^3-2\gamma^2-\gamma+3)}{(\gamma-1)^2(\gamma+1)}e^{-\lambda t}.$$

4. Limit Distributions τ_∞

The following conclusion can be obtained from the regular properties of the infinite geometric progression. If the k^{th} moment of ξ_1 exist and $\gamma\in(0,1)$ then we can assume that the k^{th} moment of τ_n exists.

Let us define some characteristics of the random operating time between failures. Let the expectation be $\mathrm{E}[\xi_1]=m$ and the variance $\mathrm{V}[\xi_1]=\sigma^2$. Then from (1) it follows that $\mathrm{E}[\xi_n]=\gamma^{n-1}m$ and $\mathrm{V}[\xi_n]=\gamma^{2(n-1)}\sigma^2$. Based on these expressions it can be shown that the following statement holds for the ageing objects. The restored object spends the resource during finite time.

To demonstrate the above statement define the sum

$$\sum_{i=1}^{\infty}\mathrm{E}[\xi_i]=\sum_{i=1}^{\infty}\gamma^{i-1}m=\frac{m}{1-\gamma},(\gamma\le 1). \tag{7}$$

The above relation shows that the infinite sum of operating times between failures for the restored object is finite. Lam (2003) proved that the limiting random variable τ_∞ exists, using the fact that τ_n is supermartingale. The geometric property (2) can be generalized. Define $\mu_k^{(\eta)}=\mathrm{E}(\eta-\mathrm{E}\eta)^k$ and let $s_\infty(\gamma)=\frac{1}{1-\gamma}$. Using this definition we can prove the following property.

Property 1. The expectation and the k^{th} moment of τ_∞ exist if $\gamma\in(0,1)$ and $n\to\infty$:

$$\mathrm{E}\,\tau_\infty = m \cdot s_\infty(\gamma) = \frac{m}{1-\gamma}, \quad \mu_k^{(\tau_\infty)} = \mu_k^{(\xi)} \cdot s_\infty\left(\gamma^k\right) = \frac{\mu_k^{(\xi)}}{1-\gamma^k}, k = 2,3; \qquad (8)$$

$$\mu_4^{(\tau_\infty)} = \frac{\mu_4^{(\xi)}}{1-\gamma^4} + 6\frac{\left(\mu_2^{(\xi)}\right)^2 \gamma^2}{\left(1-\gamma^2\right)\left(1-\gamma^4\right)}; \quad \mu_5^{(\tau_\infty)} = \frac{\mu_5^{(\xi)}}{1-\gamma^5} + 10\frac{\mu_2^{(\xi)}\mu_3^{(\xi)}\left(\gamma^2+\gamma^3\right)}{\left(1-\gamma^2\right)\left(1-\gamma^3\right)\left(1-\gamma^5\right)}. \qquad (9)$$

We can take, e.g., a normally distributed random variable ξ_1 with mean m and variance σ^2, $(m > 3\sigma)$. Using properties of the normal distribution we can obtain

$$\tau_n \sim N\left(m\frac{1-\gamma^n}{1-\gamma}, \sigma^2\frac{1-\gamma^{2n}}{1-\gamma^2}\right), \ \tau_\infty \sim N\left(\frac{m}{1-\gamma}, \frac{\sigma^2}{1-\gamma^2}\right). \qquad (10)$$

It is easy to see that if $m > 3\sigma$ then $\frac{m}{1-\gamma} > 3\frac{\sigma}{\sqrt{1-\gamma^2}}$.

Take $\xi_1 \sim \delta_m$ with singular point m. In this case τ_n will be a singular variable with singular point $m(1-\gamma^n)/(1-\gamma)$. Then the instance of failure will be a geometric progression subsum.

$$\tau_n = g(n) = m \cdot s_n(\gamma) = m\frac{\gamma^n - 1}{\gamma - 1}. \qquad (11)$$

and

$$\tau_n \sim \delta_{m\frac{1-\gamma^n}{1-\gamma}}, \ \tau_\infty \sim \delta_{\frac{m}{1-\gamma}}. \qquad (12)$$

Studies showed that if $\gamma \in (0,1)$ then the process realizations τ_n for the irregular case, can be described sufficiently by (11) with accuracy up to a constant (see Fig. 1). Here, $\xi_1 \sim N\left(2, 0.25\right)$ with process ratio $\gamma = 0.8$ and realization volume $k = 25$. We can obtain useful information by studying the correlation between τ_n and τ_m. The correlation coefficient for the regular restoration process can be written as

$$\rho_{\tau_n,\tau_m} = \frac{\mathrm{V}\,\tau_{n\wedge m}}{\sqrt{\mathrm{V}\,\tau_n \mathrm{V}\,\tau_m}} = \frac{m}{n} \wedge \frac{n}{m}, \quad \text{where } a \wedge b = \min(a,b). \qquad (13)$$

If m is constant and $n \to \infty$, then $\rho_{\tau_n,\tau_m} \to 0$. So, if n is large it is practically impossible to predict more or less, exactly τ_n when τ_m is known.

Using Property 1 we can obtain Property 2.

Property 2. For the geometric process with finite variance $V\xi_1 < \infty$ correlation is defined as

$$\rho_{\tau_n,\tau_m} = \frac{\left|1-\gamma^{2(n\wedge m)}\right|}{\sqrt{\left(1-\gamma^{2n}\right)\left(1-\gamma^{2m}\right)}} = \sqrt{\frac{1-\gamma^{2m}}{1-\gamma^{2n}}} \wedge \sqrt{\frac{1-\gamma^{2n}}{1-\gamma^{2m}}}. \qquad (14)$$

Let $\gamma \in (0,1)$. If $n \to \infty$ the correlation is nonzero, if m is nonzero

$$\lim_{n\to\infty} \rho_{\tau_n,\tau_m} = 1-\gamma^m. \qquad (15)$$

If m is big, then a sufficiently exact optimal prediction of τ_n can be obtained. If $\gamma > 1$ then the correlation tends to 0 (as in case of RP process).

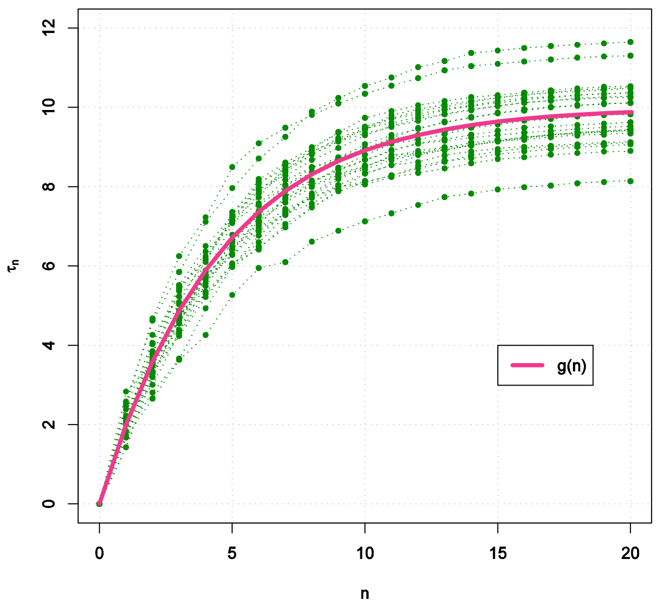

Figure 1: Process path function τ_n, $(0 < \gamma < 1)$

5. IF and RF Asymptotic Behavior

In Braun et. al. (2005) it is proved that the RF ($\gamma < 1$) in each interval of time is infinite, a result that was generalized by Finkelstein (2010). In Braun et. al (2005) it was considered that the governing lifetime GDF $F(t)$ is absolutely continuous, strictly positive, and strictly increasing for all $t > 0$. In Finkelstein (2008) the main requirement is that $F(\varepsilon) > 0$ for all $\varepsilon > 0$ and $\gamma < 1$. The condition $F(\varepsilon) > 0$ holds for numerous distributions in reliability applications such as exponential, gamma and Weibull. Braun et. al (2005) established that if $F(\varepsilon) = 0$ for some $\varepsilon > 0$, then RF can be finite for small *t*. Note though that this proof can be modified to show that RF is infinite for all $t > \tau$ where τ is such that $F(\tau) > 0$.

We are interested in the asymptotic behavior of RF & IF when $F(\varepsilon) = 0$ for some $\varepsilon > 0$. The limit distribution exists for $\gamma \in (0,1)$ hence $\Omega(x) = \sum_{n=1}^{\infty} F_n(x)$ tends to $+\infty$ as *x* attained $\tau_\infty^{(\alpha)}$ (the α - percentage point of the limit distribution).

Define the set $X_{\varepsilon}^{(f)} = \left\{x : f_{\tau_\infty}(x) > \varepsilon\right\}$ for $\varepsilon > 0$. Let $x_\varepsilon = \inf X_{\varepsilon}^{(f)}$. The series $\sum_{n=1}^{\infty} f_n(x_\varepsilon)$ is divergent for every fixed $\varepsilon > 0$. Hence the following limit is the vertical asymptote for IF and RF:

$$\underline{\tau_\infty} = \lim_{\varepsilon \to 0+} x_\varepsilon. \tag{16}$$

In Fig. 2 the summarized densities for the normal operating time with expectation $m = 2$, variance $\sigma^2 = 0.25$ and $\gamma = 0.75$ are depicted.

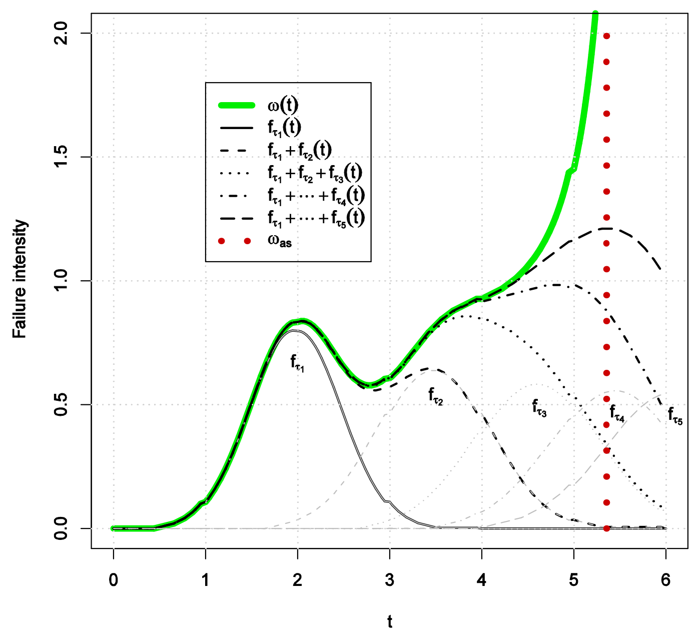

Figure 2: IF and densities $f_{\tau_n}(t)$, $\xi \sim N\left(2, 0.25\right)$

To find the IF we calculate the finite sum $\omega_n(t) = \sum_{1}^{n} f_k(t)$ for $n = 500$. The value $\frac{m}{1-\gamma} - 3.5\frac{s}{\sqrt{1-\gamma^2}} = 5.375$ was selected as asymptotic. Note that the real one should be smaller. RF will have the same vertical asymptote since IF is the derivative of RF.

Note that power indices of exponent differ by a constant equal to γ. Now, we try to obtain the asymptotic solution for the given index considering the cases $\gamma > 1$ and $0 < \gamma < 1$.

1. The cumulative intensity function increases by one unit when the object fails. An average time increment in which one failure occurs is equal to the average operating time. For the first operating time we have $E\xi = m$, for the second γm, for the third $\gamma^2 m$ etc. In general the average duration of the n^{th} operating time is equal to $\gamma^{n-1}m$.

The ratio of the averages of the $(n+1)^{\text{th}}$ and the n^{th} operating times is constant and equal to γ.

Consider the function $g(t) = b\left(a^t - 1\right)$ and the ratio

$$\frac{g(n+1)-g(n)}{g(n)-g(n-1)} = \frac{a^{n+1}-a^n}{a^n-a^{n-1}} = a.$$

Take $\gamma = a$ and $b\left(a^1 - 1\right) = m$. Then $g(t) = \frac{m}{\gamma-1}\left(\gamma^t - 1\right)$. The inverse of g(t) possesses all properties of the cumulative intensity function and it is the asymptotic solution for the cumulative intensity function:

$$\Omega(t) \sim g^{-1}(t) = \log_\gamma\left(1 + \frac{\gamma-1}{m}t\right). \tag{17}$$

The asymptotic solution for the intensity can be written as

$$\omega(t) \sim \left(g^{-1}(t)\right) = \frac{\gamma-1}{\ln\gamma\left(m+(\gamma-1)t\right)} \xrightarrow[t\to\infty]{} 0. \tag{18}$$

This case corresponds to the system functioning at the stage of infant mortality (rejuvenescent system).

2. Let $0 < \gamma < 1$. Similar reasoning leads to the function $g(t) = \frac{m}{1-\gamma}\left(1-\gamma^t\right)$. The inverse function possesses all properties of cumulative intensity function

$$\Omega(t) \sim g^{-1}(t) = \log_\gamma\left(1 - \frac{1-\gamma}{m}t\right). \tag{19}$$

The asymptotic solution for IF is defined as

$$\omega(t) \sim \left(g^{-1}(t)\right)' = \frac{1-\gamma}{\ln\gamma\left(m-(1-\gamma)t\right)} \xrightarrow[t\to\frac{m}{1-\gamma}-0]{} \infty. \tag{20}$$

The second case allows describing the system with incomplete recovery.

For the given model the following result is obtained. The intensity tends to infinity on the bounded interval of the time which is equal to $\frac{m}{1-\gamma}$. This means that the system practically spends the resource and after recovery fails again when the time tends to the given time instant.

If $\xi_1 \sim \delta_m$ then using (12) we obtain the following result (Finkelstein, 2010)

$$\Omega(x) = \begin{cases} 0, & x < m; \\ k, & x \in \left[m\dfrac{1-\gamma^k}{1-\gamma}, m\dfrac{1-\gamma^{k+1}}{1-\gamma}\right), k \in \mathbb{N}. \end{cases} \tag{21}$$

RF will increase each time by 1 at points of failures, i.e. $\Omega(\tau_n) = n.$ RF is constant on intervals $\left[\tau_n, \tau_{n+1}\right)$. We define a monotonic function of trend for RF, namely $\tilde{\Omega}(t)$. This function must coincide with the RF graph at the points τ_n: $\tilde{\Omega}(\tau_n) = \Omega(\tau_n)$.

It follows from (18) that the inverse function of $g(t) = ms_t(\gamma) = m\left(1-\gamma^t\right)/\left(1-\gamma\right)$ is:

$$\tilde{\Omega}(t) = g^{-1}(t) = \log_\gamma\left(1+\frac{\gamma-1}{m}t\right). \qquad (22)$$

This expression can be compared with (12). Hence we can suppose that the RF trend can be defined by (19) in the case of a random process.

6. Geometric of Renewal Function and Function Intensities

In Section 5, we mentioned that the RF is infinite in each time interval if $F(\varepsilon) > 0$ for all $\varepsilon > 0$ and $\gamma < 1$. If $F(\varepsilon) = 0$ for all $\varepsilon > 0$, then RF and IF for $\gamma \in (0,1)$, have a vertical asymptote and are defined on some bounded set. For a complete description of all properties of the geometric process ($\gamma \in (0,1)$) we introduce some additional features.

Definition 2. Let ξ_t be a geometric process with the ratio $\gamma \neq 1$. *Geometric of renewal function (GRF)* is a function of the real variable:

$$\Upsilon(t) = \mathrm{E}\frac{\gamma^{\xi_t+1}-1}{\gamma-1}, \quad t \in [0,\infty). \qquad (23)$$

Geometric of (failure) intensity function (GIF) is a function of the real variable

$$\nu(t) = \frac{d}{dt}\Upsilon(t), \quad t \in [0,\infty). \qquad (24)$$

Property 3. GRF and GIF can be written as

$$\Upsilon(t) = \sum_0^\infty \gamma^n F_n(t), \quad \nu(t) = \sum_0^\infty \gamma^n f_n(t). \qquad (25)$$

Proof. $\mathrm{E}\frac{\gamma^{\xi_t+1}-1}{\gamma-1} = \mathrm{E}\sum_0^{\xi_t}\gamma^i = \mathrm{E}\sum_0^\infty \gamma^i \mathrm{I}\{\xi_t \geq i\} = \mathrm{E}\sum_0^\infty \gamma^i \mathrm{I}\{\tau_i \leq t\} = \sum_0^\infty \gamma^i \mathrm{P}(\tau_i \leq t)$.□

Using the L' Hospital rule we obtain

$$\lim_{\gamma\to 1}\Upsilon(t) = \mathrm{E}(\xi_t+1) = \Omega(t)+\theta(t), \quad \lim_{\gamma\to 1}\nu(t) = \omega(t)+\delta(t), \qquad (26)$$

where $\theta(t)$ is the Heaviside function and $\delta(t)$ is the Dirac delta-function. We can now derive analogues of the renewal equations for the new features:

$$\Upsilon(x) = \theta(x) + \gamma\int_0^x \sum_{n=2}^\infty \gamma^{n-1}F_{n-1}\left(\frac{x-u}{\gamma}\right) f(u)du =$$
$$= \theta(x) + \gamma\int_0^x \Omega\left(\frac{x-u}{\gamma}\right) f(u)du = \theta(x) + \gamma\int_0^{x/\gamma}\omega(u)F(x-\gamma u)du. \qquad (27)$$

$$\nu(x) = \delta(x) + \int_0^x \sum_{n=2}^\infty \gamma^{n-1} f_{n-1}\left(\frac{x-u}{\gamma}\right) f(u)du =$$
$$= \delta(x) + \int_0^x \nu\left(\frac{x-u}{\gamma}\right) f(u)du = \delta(x) + \gamma\int_0^{x/\gamma}\nu(u) f(x-\gamma u)du.$$

The graph of GIF $\nu(t)$ without the first term (the function $\delta(t)$) is shown in Fig.3.

Also graphs of terms and resulting sums appear in the figure. The r.v. ξ_1 is simulated by the normal law $N(2,0.25)$ and the process ratio $\gamma = 0.75$. To obtain $v(t)$ the finite sum $v_n(t) - \delta(t) = \sum_1^n \gamma^k f_k(t)$ was calculated. We use *n=500* for the calculations. The area of localization for the density function of τ_∞ - $\left[\underline{\tau_\infty}, \overline{\tau_\infty}\right]$ is denoted by dotted lines. This area corresponds to the support of the distribution.

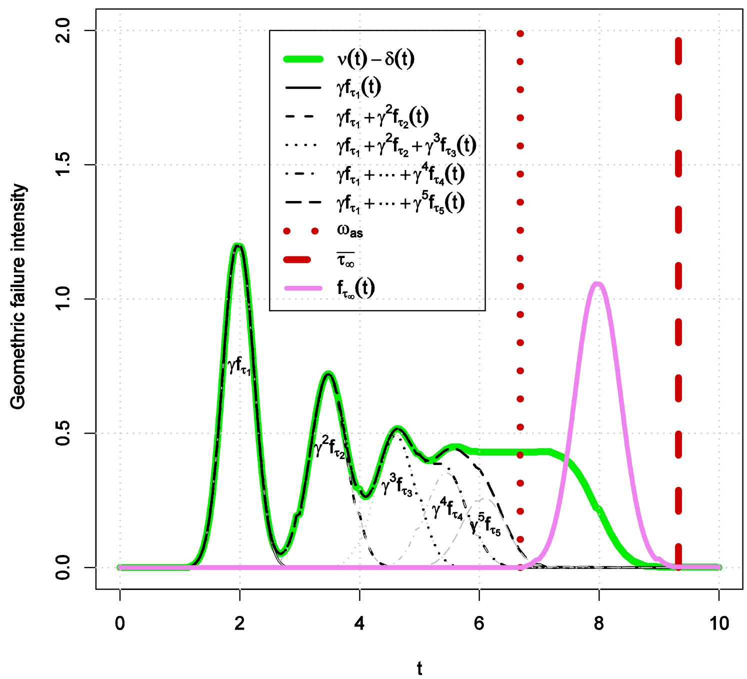

Figure 3: GIF, $\left(\gamma \in (0,1)\right)$

We take $\left(\underline{\tau_\infty}; \overline{\tau_\infty}\right) \approx \dfrac{m}{1-\gamma} \pm 3.5\dfrac{s}{\sqrt{1-\gamma^2}} = (6.677, 9.323)$. The distribution of limit point is concentrated in this area.

Bursts of $v(t)$ occur in the modes of the density (maximum points).

This is clearly seen at the beginning of observation. The first mode is equal to 2, the second $2 + 2 \cdot 0.75 = 3.5$, etc. Then the gradual right shift of maximum points of $v(t)$ relative to the modes of the density $f_k(t)$ takes place.

This graph looks like the RF graph for the normal renewal process. However there is a critical distinction. A gradual GIF decrease occurs after completion of the oscillating process. This process begins at the α -percentage point of the distribution of τ_∞. As it seen from Fig. 3 the GIF goes down to zero at the $1-\alpha$ -percentage point.

The geometric process is completed and the system fails with probability one.
Note that such findings can be obtained only from the graph of $\nu(t)$. The IF graph asymptotically tends to ∞ as $t \to \underline{\tau_\infty}$. Hence the graph of $\nu(t)$ is more informative.
The graph of GRF $\Upsilon(t)$ without the first term (the function $\theta(t)$) is shown in Fig.4.

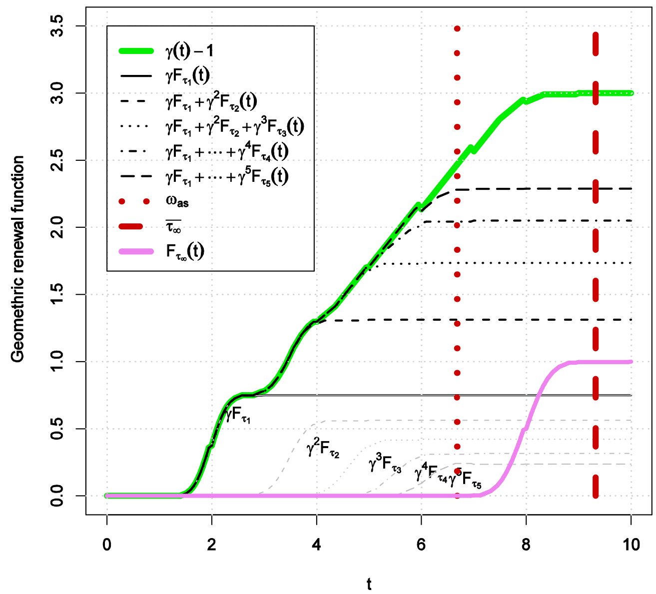

Figure 4: GRF, $(\gamma \in (0,1))$

Also graphs of terms and resulting sums appear in the figure. The input parameters of the model are the same as the ones used before. To obtain $\Upsilon(t)$ the finite sum $\Upsilon_n(t) - \theta(t) = \sum_1^n \gamma^k F_k(t)$ was calculated. We used again *n=500*.

GRF graph looks like the RF graph for the normal renewal process. However there is a critical distinction. After the completion of a linearly increasing oscillating process (at the α -percentage point of the distribution of τ_∞), the graph is almost linear at first and then gradually becomes constant $1/(1-\gamma)$ (at the $1-\alpha$ -percentage point). This looks like ECG. The geometric process is completed and the system fails with probability 1.

The GIF behavior when $\gamma > 1$ is of interest. Fig. 5 presents the summarized densities multiplied by the corresponding power γ and the summing result (GIF). The r.v. ξ_1 is simulated by the normal law $N(2, 0.25)$ with process ratio $\gamma = 1.25$.
Bursts of $\Upsilon(t)$ occur in the modes (maximum points). This is clearly seen at the beginning of observation. The first mode is equal to 2, the second $2 + 2{\cdot}1.25 = 4.5$ etc. The maximum points of $\Upsilon(t)$ coincide approximately with the modes of $f_k(t)$. The

oscillatory process is not completed. The period of oscillations gradually increases and the amplitude tends to a constant.

For example, if $\xi_1 \sim N\left(a,\sigma^2\right)$, then $\tau_n \sim N\left(a_n,\sigma_n^2\right)$, where $a_n = \mathrm{M}\tau_n = a\left(\gamma^n - 1\right)/\gamma - 1$ and $\sigma_n^2 = \operatorname{var}\tau_n = \sigma^2\left(\gamma^{2n}-1\right)\big/\gamma^2 - 1$. In this case, each term of GIF intensity tends to the next constant in the point of maximum

$$\gamma^n f_n(a_n) = \frac{1}{\sqrt{2\pi}\sigma}\sqrt{\frac{\gamma^2-1}{\gamma^{2n}-1}}\gamma^n \xrightarrow[n\to\infty]{} \frac{\sqrt{\gamma^2-1}}{\sqrt{2\pi}\sigma}.$$

The difference between two nearest maximum points will increase indefinitely (since $\gamma > 1$). $\mathrm{M}\tau_{n+1} - \mathrm{M}\tau_n = a\gamma^n \xrightarrow[n\to\infty]{} \infty.$

In summary, we conclude that the use of geometric characteristics for the geometric process has some advantages over traditional RF and IF.

We formulate the basic properties of the newly introduced geometric characteristics based on the analysis of the graphs.

Property 4. GRF $\mathrm{Y}(t)$ does not decrease. GIF is nonnegative: $\nu(t) \geq 0$.

$$\lim_{t\to\infty}\mathrm{Y}(t) = \int_0^\infty \nu(t)dt = \frac{1}{1-\gamma},\ \lim_{t\to\infty}\nu(t) = 0.$$

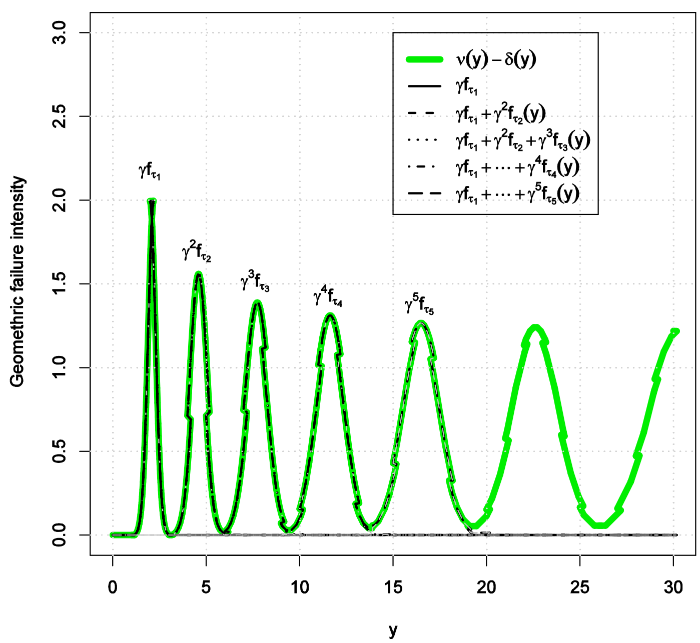

Figure: 5. GIF, $(\gamma > 1)$

7. Conclusion

In the present report we considered a reliability analysis method for renewal objects taking into account incomplete repair. We obtained the asymptotic solution for the intensity of the geometric process model. Reliability characteristics of the geometric process model for various distributions and various parameters were calculated. We obtained such characteristic as the correlation between the recovery times τ_n and τ_m. We derived asymptotic properties for IF and RF when $\gamma < 1$. Also, we introduced and analyzed new measures of the geometric process, namely GIF and GRF.

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