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SOME CONSTRUCTIONS OF BALANCED INCOMPLETE BLOCK DESIGN WITH NESTED ROWS AND COLUMNS

Vineeta Singh* and Keerti Jain**
*Department of Statistics, Institute of Social Science,
Dr. B.R.Ambedkar University, Agra (India)
E Mail: vineeta20012002@ yahoo.co.in.
**Department of Applied Science,
Echelon Institute of Technology, Faridabad, Haryana, India.
E Mail: keerti2401@rediffmail.com.
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Abstract

Many researchers have given methods of construction of balanced incomplete block designs with nested rows and columns. In this paper some new methods of construction of balanced incomplete block designs with nested rows and columns are developed. The paper concludes with a numerical illustration and the appendix consists of parameters of some BIB-RC designs with their efficiencies and efficiency factors.

Key words: Balanced incomplete block design; Nested row column design.

1. Introduction

If v treatments are arranged in b blocks of size $k = pq < v$, where each block is composed of p rows and q columns, such an arrangement is said to be incomplete block design with nested rows and columns, This design is said to be balanced if :

(i) A treatment occurs atmost once in each block,
(ii) Each treatment occurs in r blocks, and
(iii) Given a pair of symbols (i, j),

$$(p-1)\lambda_{r(i,j)} + (q-1)\lambda_{c(i,j)} - \lambda_{e(i,j)} = \lambda^* \text{ (say)} \quad (1.1)$$

occurs in the same row, same column and elsewhere respectively and λ^* is a constant independent of the pair of treatments chosen and satisfies,

$$\lambda^* (v-1) = r(p-1)(q-1). \quad (1.2)$$

These designs are denoted by BIB-RC (v, b, r, p, q, λ^*).

Singh and Dey (1979) first introduced a balanced incomplete block design with nested rows and columns and they have given several methods of construction and examples. Since then, several method of construction were given by Street (1981), Agrawal and Prasad (1982, 1983) , Jimbo and Kuriki (1983), Cheng (1986), Uddin and Morgan (1990), Mukerjee and Gupta (1991), Uddin(1995), Morgan (1996), Takaaki Hishida and Jimbo (2002),etc.

Agrawal and Prasad (1982, 1983) presented some methods of construction of balanced incomplete block design with nested rows and columns using the concepts of method of differences. Further, Uddin and Morgan (1990), Sreenath(1991) also worked on BIB-RC design, using the same concept and gave some new constructions of BIB-RC design. Uddin (1992) presented four new infinite series of designs, in which each of the row, column, and block component designs is a balanced incomplete block design.

Udddin (1995) presented a recursive method for the construction of balanced incomplete block designs with nested rows and columns. Morgan and Uddin (1997) constructed two infinite series, which are universally optimum for the analysis with recovery of row and column information, using the know series. Chauhan and Martin (2001) developed optimal nested row-column designs for block of size 2×2 under dependence. Gupta, Voss, and Prasad (2001) presented a paper on optimal nested row and column design. One of these series achieves orthogonality with just $v-1$ replicates of v treatments, fewer than required by latin square. Hishida and Jimbo (2002) presented a paper in which the constructions given, generate BIBD-RCs having the same parameters as several series due to Uddin and Morgan (1990) as their special cases. Uddin and Morgan developed a method of construction of class of universally optimal structurally incomplete row column design. In 2003, they presented optimal row-column design for two treatments. Again in 2005, Uddin developed a method of construction of a universally optimal structurally balanced row-column design with some empty nodes. In 2007, he presented neighbor properties of some classes of BIB-RC design and their efficiencies for correlated errors.

In this paper some simple methods of construction of BIB-RC design are given. These methods are particular cases of the theorem 4.2.1. of Sreenath (1999), which states that the existence of a BIB design with parameters v, b, r, k, λ and a BIB-RC design with parameters k, b^*, r^*, u, w, λ^* implies the existence of BIB-RC design with parameters v, bb^*, rr^*, u, w, $\lambda\lambda^*$. In these designs blocks of BIB-RC design with parameters k, b^*, r^*, u, w, λ* are superimposed on the blocks of BIB design with parameter v, b, r, k, λ to form a BIB-RC design with parameters v, bb^*, rr^*, u, w, $\lambda\lambda^*$. Here a small BIB-RC design, i.e. BIB-RC design with small v is used to form a BIB-RC design with large v.

2. Construction

Theorem 1: Let $v=(p^n)^2$ be a square of a prime power, then for p = 2, there exists a BIB-RC design with parameters:

$$v=(p^n)^2, b=p^n[(p^n)^2-1], r=(p^n)^2-1, u=p^{n-1}, w=2, \lambda^*=(p^{n-1}-1)$$

where u and w are number of treatments in rows and columns of a block respectively.

Proof :- Consider a geometry EG (2, p^n) , then for p = 2, there exist $v=(p^n)^2$ points and $v=(p^n)^2+p^n$ lines in EG (2, p^n) plane. Let us consider points as treatments and lines as blocks, i.e. points on one line are elements of the blocks, then we can always construct a BIBD [Bose(1901)] with parameters:

$$v=(p^n)^2, b=(p^n)^2+p^n, r=p^n+1, k=p^n, \lambda=1.$$

Now, blocks of this BIBD, can be arranged in $k=uw$ treatments, where $u=p^{n-1}$, $w=2$ in such a manner that the condition (i) and (ii) of BIB-RC design is satisfied. Hence the Balanced Incomplete Block with nested rows and columns exists, with parameters $v=(p^n)^2$, $b=p^n[(p^n)^2-1]$, $r=(p^n)^2-1$, $u=p^{n-1}$, $w=2$, $\lambda^*=(p^{n-1}-1)$.

Example 1: In geometry *EG* (2, 2^2), there are 16 points and 20 lines, the correspondence between points and treatments are given below:

(0, 0) → 1; (0, 1) → 2; (0, t) → 3; (0, t^2) → 4;
(1, 0) → 5; (1, 1) → 6; (1, t) → 7; (1, t^2) → 8;
(t, 0) → 9; (t, 1) → 10; (t, t) → 11; (t, t^2) → 12;
(t^2, 0) → 13; (t^2, 1) → 14; (t^2, t) → 15; (t^2, t^2) → 16.

Now, the blocks of BIBD (16, 20, 5, 4, 1) constructed by using the lines of *EG* (2, 2^2) are as follows:

(1, 5, 9, 13); (2, 6, 10, 14); (3, 7, 11, 15); (4, 8, 12, 16); (1, 6, 11, 16);
(2, 5, 12, 15); (3, 8, 9, 14); (4, 7, 10, 13); (1, 7, 12, 15); (2, 8, 11, 13);
(3, 5, 10, 16); (4, 6, 9, 15); (1, 8, 11, 15); (2, 7, 9, 16); (3, 6, 12, 14);
(4, 5, 11, 14); (1, 2, 3, 4); (5, 6, 7, 8); (9, 10, 11, 12); (13,14, 15, 16).

Now, treatments of every block of this BIBD are arranged in blocks of BIB-RC design with parameter $v = 16, b = 60, r = 15, u = 2, w = 2 \text{ and } \lambda^* = 1$, having two rows and two columns in such a way that equation (1.1) and (1.2) are satisfied. The obtained BIB-RC design is given below:

$$\begin{pmatrix}1 & 9\\5 & 13\end{pmatrix}\begin{pmatrix}1 & 13\\9 & 5\end{pmatrix}\begin{pmatrix}1 & 5\\13 & 9\end{pmatrix} \quad \begin{pmatrix}3 & 10\\5 & 16\end{pmatrix}\begin{pmatrix}3 & 16\\10 & 5\end{pmatrix}\begin{pmatrix}3 & 5\\16 & 10\end{pmatrix}$$

$$\begin{pmatrix}2 & 10\\6 & 14\end{pmatrix}\begin{pmatrix}2 & 14\\10 & 6\end{pmatrix}\begin{pmatrix}2 & 6\\14 & 10\end{pmatrix} \quad \begin{pmatrix}4 & 9\\6 & 15\end{pmatrix}\begin{pmatrix}4 & 15\\9 & 6\end{pmatrix}\begin{pmatrix}4 & 6\\15 & 9\end{pmatrix}$$

$$\begin{pmatrix}3 & 11\\7 & 15\end{pmatrix}\begin{pmatrix}3 & 15\\11 & 7\end{pmatrix}\begin{pmatrix}3 & 7\\15 & 11\end{pmatrix} \quad \begin{pmatrix}1 & 11\\8 & 15\end{pmatrix}\begin{pmatrix}1 & 15\\11 & 8\end{pmatrix}\begin{pmatrix}1 & 8\\15 & 11\end{pmatrix}$$

$$\begin{pmatrix}4 & 12\\8 & 16\end{pmatrix}\begin{pmatrix}4 & 16\\12 & 8\end{pmatrix}\begin{pmatrix}4 & 8\\16 & 12\end{pmatrix} \quad \begin{pmatrix}2 & 9\\7 & 16\end{pmatrix}\begin{pmatrix}2 & 16\\9 & 7\end{pmatrix}\begin{pmatrix}2 & 7\\16 & 9\end{pmatrix}$$

$$\begin{pmatrix}1 & 11\\6 & 16\end{pmatrix}\begin{pmatrix}1 & 16\\11 & 6\end{pmatrix}\begin{pmatrix}1 & 6\\16 & 11\end{pmatrix} \quad \begin{pmatrix}3 & 12\\6 & 13\end{pmatrix}\begin{pmatrix}3 & 13\\12 & 6\end{pmatrix}\begin{pmatrix}3 & 6\\13 & 12\end{pmatrix}$$

$$\begin{pmatrix}2 & 12\\5 & 15\end{pmatrix}\begin{pmatrix}2 & 15\\12 & 5\end{pmatrix}\begin{pmatrix}2 & 5\\15 & 12\end{pmatrix} \quad \begin{pmatrix}4 & 11\\5 & 14\end{pmatrix}\begin{pmatrix}4 & 14\\11 & 5\end{pmatrix}\begin{pmatrix}4 & 5\\14 & 11\end{pmatrix}$$

$$\begin{pmatrix}3 & 9\\8 & 14\end{pmatrix}\begin{pmatrix}3 & 14\\9 & 8\end{pmatrix}\begin{pmatrix}3 & 14\\8 & 9\end{pmatrix} \quad \begin{pmatrix}1 & 3\\2 & 4\end{pmatrix}\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\4 & 3\end{pmatrix}$$

$$\begin{pmatrix}4 & 10\\7 & 13\end{pmatrix}\begin{pmatrix}4 & 13\\10 & 7\end{pmatrix}\begin{pmatrix}4 & 7\\13 & 10\end{pmatrix} \quad \begin{pmatrix}5 & 7\\6 & 8\end{pmatrix}\begin{pmatrix}5 & 8\\7 & 6\end{pmatrix}\begin{pmatrix}5 & 6\\8 & 7\end{pmatrix}$$

$$\begin{pmatrix}1 & 12\\7 & 15\end{pmatrix}\begin{pmatrix}1 & 15\\12 & 7\end{pmatrix}\begin{pmatrix}1 & 7\\15 & 12\end{pmatrix} \quad \begin{pmatrix}9 & 11\\10 & 12\end{pmatrix}\begin{pmatrix}9 & 12\\11 & 10\end{pmatrix}\begin{pmatrix}9 & 10\\12 & 11\end{pmatrix}$$

$$\begin{pmatrix} 2 & 11 \\ 8 & 13 \end{pmatrix}\begin{pmatrix} 2 & 13 \\ 11 & 8 \end{pmatrix}\begin{pmatrix} 2 & 8 \\ 13 & 11 \end{pmatrix} \quad \begin{pmatrix} 13 & 15 \\ 14 & 16 \end{pmatrix}\begin{pmatrix} 13 & 16 \\ 15 & 14 \end{pmatrix}\begin{pmatrix} 13 & 14 \\ 16 & 15 \end{pmatrix}$$

Example 2: In geometry $EG\ (2,\ 2^3)$, there are 64 points and 72 lines, the correspondence between points and treatments are given below:

$(0, 0) \rightarrow 1$;	$(0, 1) \rightarrow 2$;	$(0, t) \rightarrow 3$;	$(0, t^2) \rightarrow 4$;
$(0, t^3) \rightarrow 5$;	$(0, t^4) \rightarrow 6$;	$(0, t^5) \rightarrow 7$;	$(0, t^6) \rightarrow 8$;
$(1, 0) \rightarrow 9$;	$(1, 1) \rightarrow 10$;	$(1, t) \rightarrow 11$;	$(1, t^2) \rightarrow 12$;
$(1, t^3) \rightarrow 13$;	$(1, t^4) \rightarrow 14$;	$(1, t^5) \rightarrow 15$;	$(1, t^6) \rightarrow 16$;
$(t, 0) \rightarrow 17$;	$(t, 1) \rightarrow 18$;	$(t, t) \rightarrow 19$;	$(t, t^2) \rightarrow 20$;
$(t, t^3) \rightarrow 21$;	$(t, t^4) \rightarrow 22$;	$(t, t^5) \rightarrow 23$;	$(t, t^6) \rightarrow 24$;
$(t^2, 0) \rightarrow 25$;	$(t^2, 1) \rightarrow 26$;	$(t^2, t) \rightarrow 27$;	$(t^2, t^2) \rightarrow 28$;
$(t^2, t^3) \rightarrow 29$;	$(t^2, t^4) \rightarrow 30$;	$(t^2, t^5) \rightarrow 31$;	$(t^2, t^6) \rightarrow 32$;
$(t^3, 0) \rightarrow 33$;	$(t^3, 1) \rightarrow 34$;	$(t^3, t) \rightarrow 35$;	$(t^3, t^2) \rightarrow 36$;
$(t^3, t^3) \rightarrow 37$;	$(t^3, t^4) \rightarrow 38$;	$(t^3, t^5) \rightarrow 39$;	$(t^3, t^6) \rightarrow 40$;
$(t^4, 0) \rightarrow 41$;	$(t^4, 1) \rightarrow 42$;	$(t^4, t) \rightarrow 43$;	$(t^4, t^2) \rightarrow 44$;
$(t^4, t^3) \rightarrow 45$;	$(t^4, t^4) \rightarrow 46$;	$(t^4, t^5) \rightarrow 47$;	$(t^4, t^6) \rightarrow 48$;
$(t^5, 0) \rightarrow 49$;	$(t^5, 1) \rightarrow 50$;	$(t^5, t) \rightarrow 51$;	$(t^5, t^2) \rightarrow 52$;
$(t^5, t^3) \rightarrow 53$;	$(t^5, t^4) \rightarrow 54$;	$(t^5, t^5) \rightarrow 55$;	$(t^5, t^6) \rightarrow 56$;
$(t^6, 0) \rightarrow 57$;	$(t^6, 1) \rightarrow 58$;	$(t^6, t) \rightarrow 59$;	$(t^6, t^2) \rightarrow 60$;
$(t^6, t^3) \rightarrow 61$;	$(t^6, t^4) \rightarrow 62$;	$(t^6, t^5) \rightarrow 63$;	$(t^6, t^6) \rightarrow 64$.

Now, the blocks of BIBD (64, 72, 9, 8, 1) constructed by using the lines of $EG(2,\ 2^3)$, are as follows:

(1, 9, 17, 25, 33, 41, 49, 57); (2, 10, 18, 26, 34, 42, 50, 58);
(3, 11, 19, 27, 35, 43, 51, 59); (4, 12, 20, 28, 36, 44, 52, 60);
(5, 13, 21, 29, 37, 45, 53, 61); (6, 14, 22, 30, 38, 46, 54, 62);
(7, 15, 23, 31, 39, 47, 55, 63); (8, 16, 24, 32, 40, 48, 56, 64);
(1, 10, 19, 28, 37, 46, 55, 64); (2, 9, 23, 29, 36, 48, 51, 62);
(3, 15, 17, 32, 38, 45, 50, 60); (4, 13, 24, 25, 34, 47, 54, 59);
(5, 12, 22, 26, 33, 43, 56, 63); (6, 16, 21, 31, 35, 41, 52, 58);
(7, 11, 18, 30, 40, 44, 49, 61); (8, 14, 20, 27, 39, 42, 53, 57);
(1, 11, 20, 29, 38, 47, 56, 57); (2, 15, 21, 28, 40, 43, 54, 58);
(3, 9, 24, 30, 37, 42, 52, 59); (4, 13, 17, 26, 39, 46, 51, 60);
(5, 14, 18, 25, 35, 48, 55, 61); (6, 16, 23, 27, 33, 44, 50, 62);
(7, 10, 22, 32, 36, 41, 53, 63); (8, 12, 19, 31, 34, 45, 49, 64);
(1, 12, 21, 30, 39, 48, 49, 58); (2, 13, 20, 32, 35, 46, 50, 57);
(3, 16, 22, 29, 34, 44, 51, 63); (4, 9, 18, 31, 38, 43, 52, 61);
(5, 10, 17, 27, 40, 47, 53, 60); (6, 15, 19, 25, 36, 42, 54, 64);
(7, 14, 24, 28, 33, 45, 55, 59); (8, 15, 18, 29, 33, 48, 54, 60);
(1, 14, 23, 32, 33, 42, 51, 60); (2, 16, 19, 30, 34, 41, 55, 61);
(3, 13, 18, 28, 35, 47, 49, 64); (4, 15, 22, 27, 36, 45, 56, 57);
(5, 11, 24, 31, 37, 44, 54, 58); (6, 9, 20, 26, 38, 48, 53, 63);
(7, 12, 17, 29, 39, 43, 50, 62); (8, 10, 21, 25, 40, 46, 52, 59);
(1, 15, 24, 25, 34, 43, 52, 61); (2, 11, 22, 26, 33, 47, 53, 60);
(3, 10, 20, 27, 39, 41, 56, 62); (4, 14, 19, 28, 37, 48, 49, 58);
(5, 16, 23, 29, 36, 46, 50, 57); (6, 12, 18, 30, 40, 45, 55, 59);

(7, 9, 12, 31, 35, 42, 54, 64); (8, 13, 17, 32, 38, 44, 51, 63);
(1, 16, 17, 27, 36, 45, 54, 63); (2, 14, 18, 25, 39, 44, 56, 59);
(3, 12, 19, 31, 33, 48, 54, 63); (4, 11, 20, 29, 40, 41, 50, 63);
(5, 15, 21, 28, 38, 42, 49, 58); (6, 10, 22, 32, 37, 47, 51, 57);
(7, 13, 23, 27, 34, 46, 56, 60); (8, 9, 24, 30, 36, 43, 55, 58);
(1, 2, 3, 4, 5, 6, 7, 8); (9, 10, 11, 12, 13, 14, 15, 16);
(17, 18, 19, 20, 21, 22, 23, 24); (25, 26, 27, 28, 29, 30, 31, 32);
(33, 34, 35, 36, 37, 38, 39, 40); (41, 42, 43, 44, 45, 46, 47, 48);
(49, 50, 51, 52, 53, 54, 55, 56); (57, 58, 59, 60, 61, 62, 63, 64);

Now, treatments of every block of this BIBD, are arranged in blocks of BIB-RC design (64, 504, 63, 4, 2, 3) having four rows and two columns in such a way that equation (1.1) and (1.2) are satisfied.

Some blocks of BIB-RC design obtained from first block of BIBD are given below:

$$\begin{pmatrix} 1 & 9 \\ 42 & 33 \\ 17 & 57 \\ 25 & 49 \end{pmatrix} \quad \begin{pmatrix} 1 & 17 \\ 9 & 57 \\ 25 & 33 \\ 49 & 41 \end{pmatrix} \quad \begin{pmatrix} 1 & 25 \\ 17 & 41 \\ 9 & 49 \\ 57 & 33 \end{pmatrix} \quad \begin{pmatrix} 1 & 33 \\ 25 & 49 \\ 17 & 41 \\ 9 & 57 \end{pmatrix} \quad \begin{pmatrix} 41 & 1 \\ 4 & 49 \\ 17 & 33 \\ 25 & 57 \end{pmatrix} \quad \begin{pmatrix} 49 & 1 \\ 25 & 57 \\ 9 & 41 \\ 17 & 33 \end{pmatrix}$$

$$\begin{pmatrix} 57 & 1 \\ 17 & 33 \\ 25 & 49 \\ 9 & 41 \end{pmatrix}$$

Other patterns can be obtained, using similar pattern**.**

Theorem 2: Let $v = (p^n)^2$ be a square of a prime power, then for $p = 3$, there exists a BIB-RC design with parameters:

$$v = (p^n)^2, b = p[(p^n)^2 - 1], r = (p^n)^2 - 1,$$
$$u = p^{n-1}, w = 3 and \lambda^* = 2(p^{n-1} - 1).$$

where u and w are number of treatments in rows and columns of a block respectively.

***Proof:* -** Consider a geometry *EG(2,* p^n*),* for *p = 3*, there exists $v = (p^n)^2$ points and $v = (p^n)^2 + p^n$ lines in *EG*(2, p^n) plane, let us consider points as treatments and lines as blocks, i.e. points on one line are elements of the blocks, then we can always construct a BIBD, with parameters:

$$v = (p^n)^2, b = (p^n)^2 + p^n, r = p^n + 1, k = p^n, \lambda = 1.$$

Now, blocks of this BIBD, can be arranged in $k = uw$ treatments.

The condition (i) and (ii) for the existence of BIB-RC design are satisfied for the resulting arrangement of v treatments arranged in b blocks, each of $k = uw$ treatments and

Since the value of $\lambda_{\alpha(i,j)} = 2, \lambda_{\beta(i,j)} = (p^{n-1} - 1) \,\&\, \lambda_{\gamma(i,j)} = 2(p^{n-1} - 1),$ hence the equation (1.1) is satisfied.

Also, $v = (p^n)^2, r = (p^n)^2 - 1, u = p^{n-1}, w = 3, \lambda^* = 2(p^{n-1} - 1)$, satisfy equation (1.2). Hence the Balanced Incomplete Block with nested rows and columns exists.

Example 3: In geometry *EG (2, 3^2)*, there are 81 points and 90 lines, the correspondence between points and treatments are given below:

$(0, 0) \rightarrow 1$;	$(0, 1) \rightarrow 2$;	$(0, t) \rightarrow 3$;	$(0, t^2) \rightarrow 4$;
$(0, t^3) \rightarrow 5$;	$(0, t^4) \rightarrow 6$;	$(0, t^5) \rightarrow 7$;	$(0, t^6) \rightarrow 8$;
$(0, t^7) \rightarrow 9$;	$(1, 0) \rightarrow 10$;	$(1, 1) \rightarrow 11$;	$(1, t) \rightarrow 12$;
$(1, t^2) \rightarrow 13$;	$(1, t^3) \rightarrow 14$;	$(1, t^4) \rightarrow 15$;	$(1, t^5) \rightarrow 16$;
$(1, t^6) \rightarrow 17$;	$(1, t^7) \rightarrow 18$;	$(t, 0) \rightarrow 19$;	$(t, 1) \rightarrow 20$;
$(t, t) \rightarrow 21$;	$(t, t^2) \rightarrow 22$;	$(t, t^3) \rightarrow 23$;	$(t, t^4) \rightarrow 24$;
$(t, t^5) \rightarrow 25$;	$(t, t^6) \rightarrow 26$;	$(t, t^7) \rightarrow 27$;	$(t^2, 0) \rightarrow 28$;
$(t^2, 1) \rightarrow 29$;	$(t^2, t) \rightarrow 30$;	$(t^2, t^2) \rightarrow 31$;	$(t^2, t^3) \rightarrow 32$;
$(t^2, t^4) \rightarrow 33$;	$(t^2, t^5) \rightarrow 34$;	$(t^2, t^6) \rightarrow 35$;	$(t^2, t^7) \rightarrow 36$;
$(t^3, 0) \rightarrow 37$;	$(t^3, 1) \rightarrow 38$;	$(t^3, t) \rightarrow 39$;	$(t^3, t^2) \rightarrow 40$;
$(t^3, t^3) \rightarrow 41$;	$(t^3, t^4) \rightarrow 42$;	$(t^3, t^5) \rightarrow 43$;	$(t^3, t^6) \rightarrow 44$;
$(t^3, t^7) \rightarrow 45$;	$(t^4, 0) \rightarrow 46$;	$(t^4, 1) \rightarrow 47$;	$(t^4, t) \rightarrow 48$;
$(t^4, t^2) \rightarrow 49$;	$(t^4, t^3) \rightarrow 50$;	$(t^4, t^4) \rightarrow 51$;	$(t^4, t^5) \rightarrow 52$;
$(t^4, t^6) \rightarrow 53$;	$(t^4, t^7) \rightarrow 54$;	$(t^5, 0) \rightarrow 55$;	$(t^5, 1) \rightarrow 56$;
$(t^5, t) \rightarrow 57$;	$(t^5, t^2) \rightarrow 58$;	$(t^5, t^3) \rightarrow 59$;	$(t^5, t^4) \rightarrow 60$;
$(t^5, t^5) \rightarrow 61$;	$(t^5, t^6) \rightarrow 62$;	$(t^5, t^7) \rightarrow 63$;	$(t^6, 0) \rightarrow 64$;
$(t^6, 1) \rightarrow 65$;	$(t^6, t) \rightarrow 66$;	$(t^6, t^2) \rightarrow 67$;	$(t^6, t^3) \rightarrow 68$;
$(t^6, t^4) \rightarrow 69$;	$(t^6, t^5) \rightarrow 70$;	$(t^6, t^6) \rightarrow 71$;	$(t^6, t^7) \rightarrow 72$;
$(t^7, 0) \rightarrow 73$;	$(t^7, 1) \rightarrow 74$;	$(t^7, t) \rightarrow 75$;	$(t^7, t^2) \rightarrow 76$;
$(t^7, t^3) \rightarrow 77$;	$(t^7, t^4) \rightarrow 78$;	$(t^7, t^5) \rightarrow 79$;	$(t^7, t^6) \rightarrow 80$;
$(t^7, t^7) \rightarrow 81$.			

Now, the blocks of BIBD (81, 90, 10, 9, 1) constructed by using the lines of *EG* (2, 3^2) are as follows:

(1, 10, 19, 28, 37, 46, 55, 64, 73);	(2, 11, 20, 29, 38, 47, 56, 65, 74);
(3, 12, 21, 30, 39, 48, 57, 66, 75);	(4, 13, 22, 31, 40, 49, 58, 67, 76);
(5, 14, 23, 32, 41, 50, 59, 68, 77);	(6, 15, 24, 33, 42, 51, 60, 69, 78);
(7, 16, 25, 34, 43, 52, 61, 70, 79);	(8, 17, 26, 35, 44, 53, 62, 71, 80);
(9, 18, 27, 36, 45, 54, 63, 72, 81);	(1, 11, 21, 31, 41, 51, 61, 71, 81);
(2, 15, 27, 32, 43, 46, 58, 66, 80);	(3, 18, 25, 29, 42, 53, 55, 68, 76);
(4, 14, 20, 35, 38, 52, 63, 64, 78);	(5, 25, 24, 30, 45, 49, 62, 65, 73);
(6, 10, 26, 34, 40, 47, 59, 72, 75);	(7, 13, 19, 36, 44, 50, 57, 69, 74);
(8, 12, 23, 28, 38, 54, 60, 67, 79);	(9, 17, 22, 33, 37, 48, 56, 70, 77);

(1, 12, 22, 32, 42, 52, 62, 72, 74); (2, 18, 23, 34, 37, 49, 57, 71, 78);
(3, 16, 20, 33, 44, 46, 59, 67, 81); (4, 11, 26, 30, 43, 54, 55, 69, 77);
(5, 15, 21, 36, 40, 53, 56, 64, 79); (6, 17, 25, 31, 38, 50, 63, 66, 73);
(7, 10, 27, 35, 41, 48, 60, 65, 76); (8, 14, 19, 29, 45, 51, 58, 70, 75);
(9, 13, 24, 28, 39, 47, 61, 68, 80); (1, 13, 23, 33, 43, 53, 63, 65, 74);
(2, 14, 25, 28, 40, 48, 62, 69, 81); (3, 11, 24, 35, 37, 50, 58, 72, 75);
(4, 17, 21, 34, 45, 46, 60, 68, 74); (5, 12, 27, 31, 44, 47, 55, 70, 78);
(6, 16, 22, 29, 41, 54, 57, 67, 80); (7, 18, 26, 32, 39, 51, 57, 67, 73);
(8, 10, 20, 36, 42, 49, 61, 66, 77); (9, 15, 19, 30, 38, 52, 59, 71, 76);
(1, 14, 24, 34, 44, 54, 56, 66, 76); (2, 16, 19, 31, 39, 53, 60, 72, 77);
(3, 15, 26, 28, 41, 49, 63, 70, 74); (4, 12, 25, 36, 37, 51, 59, 65, 80);
(5, 18, 22, 35, 38, 46, 61, 69, 75); (6, 13, 20, 32, 45, 48, 55, 71, 79);
(7, 17, 23, 30, 42, 47, 58, 64, 81); (8, 11, 27, 33, 40, 52, 57, 68, 73);
(9, 10, 21, 29, 43, 50, 62, 67, 78); (1, 15, 25, 35, 45, 47, 57, 67, 77);
(2, 10, 22, 30, 44, 51, 63, 68, 79); (3, 17, 19, 32, 40, 54, 61, 65, 78);
(4, 16, 27, 28, 42, 50, 56, 71, 75); (5, 13, 26, 29, 37, 52, 60, 66, 27);
(6, 11, 23, 36, 39, 46, 62, 70, 76); (7, 14, 21, 33, 38, 49, 55, 72, 80);
(8, 18, 24, 31, 43, 48, 59, 64, 74); (9, 12, 21, 34, 41, 53, 58, 69, 73);
(1, 16, 26, 36, 38, 48, 58, 68, 78); (2, 13, 21, 35, 42, 54, 59, 70, 73);
(3, 10, 23, 31, 45, 52, 56, 69, 80); (4, 27, 19, 33, 41, 48, 62, 66, 79);
(5, 17, 20, 28, 43, 51, 57, 72, 76); (6, 14, 27, 30, 37, 53, 61, 67, 74);
(7, 12, 24, 29, 40, 46, 63, 71, 77); (8, 15, 22, 34, 39, 50, 55, 65, 81);
(9, 11, 25, 32, 44, 49, 60, 64, 75); (1, 17, 27, 29, 39, 49, 59, 69, 79);
(2, 12, 26, 33, 45, 50, 61, 64, 76); (3, 14, 22, 36, 43, 47, 60, 71, 73);
(4, 10, 24, 32, 38, 53, 57, 70, 81); (5, 11, 19, 34, 42, 47, 63, 67, 80);
(6, 18, 21, 28, 44, 52, 58, 65, 77); (7, 15, 20, 31, 37, 54, 62, 68, 75);
(8, 13, 25, 30, 41, 46, 56, 72, 78); (9, 16, 23, 35, 40, 51, 55, 66, 74);
(1, 18, 20, 30, 40, 50, 60, 70, 80); (2, 17, 24, 36, 41, 52, 55, 67, 75);
(3, 13, 27, 34, 38, 51, 62, 64, 77); (4, 15, 23, 29, 44, 48, 61, 72, 73);
(5, 10, 25, 33, 39, 49, 58, 71, 74); (6, 12, 19, 31, 43, 49, 56, 68, 81);
(7, 11, 22, 28, 45, 53, 59, 66, 78); (8, 16, 21, 32, 37, 47, 63, 69, 76);
(9, 14, 35, 31, 42, 46, 57, 65, 79); (1, 2, 3, 4, 5, 6, 7, 8, 9);
(10, 11, 12, 13, 14, 15, 16, 17, 18); (19, 20, 21, 22, 23, 24, 25, 26, 27);
(28, 29, 30, 31, 32, 33, 34, 35, 36); (37, 38, 39, 40, 41, 42, 43, 44, 45);
(46, 47, 48, 49, 50, 51, 52, 53, 54); (55, 56, 57, 58, 59, 60, 61, 62, 63);
(64, 65, 66, 67, 68, 69, 70, 71, 72); (73, 74, 75, 76, 77, 78, 79, 80, 81) .

Using this BIBD, we can obtain a BIB-RC design with parameters:

$v = 81, b = 720, r = 80, u = 3, w = 3$ and $\lambda^* = 4$,

This BIB-RC design has three rows and three columns in such a way that equation (1.1) and (1.2) are satisfied. Some blocks of BIB-RC design obtained from first block of BIBD are given below:

$$\begin{pmatrix} 1 & 28 & 37 \\ 10 & 46 & 55 \\ 19 & 64 & 73 \end{pmatrix} \begin{pmatrix} 10 & 46 & 55 \\ 1 & 28 & 37 \\ 19 & 64 & 73 \end{pmatrix} \begin{pmatrix} 37 & 55 & 73 \\ 28 & 46 & 64 \\ 1 & 10 & 19 \end{pmatrix} \begin{pmatrix} 19 & 1 & 10 \\ 64 & 28 & 46 \\ 73 & 37 & 55 \end{pmatrix}$$

$$\begin{pmatrix} 11 & 73 & 28 \\ 64 & 1 & 55 \\ 37 & 46 & 19 \end{pmatrix}\begin{pmatrix} 37 & 46 & 19 \\ 10 & 73 & 28 \\ 64 & 1 & 55 \end{pmatrix}\begin{pmatrix} 73 & 46 & 1 \\ 28 & 19 & 55 \\ 10 & 37 & 64 \end{pmatrix}\begin{pmatrix} 37 & 10 & 64 \\ 46 & 73 & 1 \\ 19 & 28 & 55 \end{pmatrix}$$

Other blocks of balanced incomplete block design can be obtained, using similar pattern.

Theorem 3 (Generalization of theorem 1 and theorem 2): If $v = (p^n)^2$ is a square of a prime power, then there exist a BIB-RC design with parameters:

$$v = (p^n)^2, b = p[(p^n)^2 - 1], r = (p^n)^2 - 1,$$
$$u = p^{n-1}, w = p and \lambda^* = (p-1)(p^{n-1} - 1).$$

where *u* and *w* are number of treatments in rows and columns of a block respectively.

***Proof:* -** For a geometry *EG*(2, p^n), there exists $v = (p^n)^2$ points and $v = (p^n)^2 + p^n$ lines in *EG* (2, p^n) plane. Let us consider points as treatments and lines as blocks, i.e. points on one line are elements of the blocks, then we can always construct a BIBD, with parameters:

$$v = (p^n)^2, b = (p^n)^2 + p^n, r = p^n + 1, k = p^n, \lambda = 1.$$

Now, blocks of this BIBD, can be arranged in $k = uw$ treatments.

The condition (i) and (ii) for the existence of BIB-RC design are satisfied for the resulting arrangement of *v* treatments arranged in *b* blocks, each of $k = uw$ treatments and
Since the value of

$\lambda_{\alpha(i,j)} = p-1, \lambda_{\beta(i,j)} = (p^{n-1} - 1)$ & $\lambda_{\gamma(i,j)} = (p-1)(p^{n-1} - 1),$ hence the equation (1.1) is satisfied.

Also, $v = (p^n)^2, r = (p^n)^2 - 1, u = p^{n-1}, w = 3, \lambda^* = 2(p^{n-1} - 1)$, satisfy equation (1.2). Hence the Balanced Incomplete Block with nested rows and columns exists.

Some more designs based on these theorems with efficiency factor and efficiency with respect to BN-RC are given in the *Appendix.*

Appendix

p	*n*	*v*	*b*	*r*	*u*	*w*	λ	Efficiency factor	Efficiency
2	2	16	60	15	2	2	1	0.2667	0.5000
2	3	64	504	63	4	2	3	0.3810	0.5000
2	4	256	4080	255	8	2	7	0.4392	0.5000
3	2	81	720	80	3	3	4	0.4500	0.6667
3	3	729	19656	728	9	3	14	0.5192	0.6667
5	2	625	15600	624	5	5	16	0.6410	0.8000

Table 1: Some designs based on Theorem 3.

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