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BAYESIAN ESTIMATION OF ERLANG DISTRIBUTION UNDER DIFFERENT PRIOR DISTRIBUTIONS

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Abstract

This paper addresses the problem of Bayesian estimation of the parameters of Erlang distribution under squared error loss function by assuming different independent informative priors as well as joint priors for both shape and scale parameters. The motivation is to explore the most appropriate prior for Erlang distribution among different priors. A comparison of the Bayes estimates and their risks for different choices of the values of the hyperparameters is also presented. Finally, we illustrate the results using a simulation study as well as by doing real data analysis.

Keywords: Bayes estimator, Posterior distribution, Squared error loss function.

1. Introduction

The queuing theory had its origin in 1909, when A.K.Erlang (1878-1929) published his fundamental paper relating to the study of congestion in telephone traffic (see Brockmeyer et. al. (1948). The literature on the theory of queues and on the diverse field of its applications has grown tremendously over the years. The analysis for such an Erlangian queue is now folklore in the queuing literature.

The Erlang distribution is the distribution of sum of exponential variates. This distribution can be expressed as waiting time and message length in telephone traffic. If the duration of individual calls are exponentially distributed then the duration of succession of calls is the Erlang distribution. The Erlang variate becomes Gamma variate when its shape parameter is an integer (for details see Evans et. al. (2000)). Harischandra and Subba Rao (1988) discussed some problems of classical inference for the Erlangian queue. Bhattacharyya and Singh (1994) obtained Bayes estimator for the Erlangian queue under two prior densities. Wiper (1998) studied for Er/M/1 and Er/M/c queues under Bayesian setup and estimated equilibrium probabilities of the queue size and waiting time distributions using conditional Monte-Carlo simulation methods. Jain (2001) discussed the problem of the change point for the inter arrival time distribution in the context of exponential families for the Ek/GIc queuing system and obtained Bayes estimates of the posterior probabilities and the positions of change from the Erlang distribution. Nair et al. (2003) studied Erlang distribution as a model for ocean wave periods and obtained different characteristics of this distribution under classical set up. Suri et al (2009) used Erlang distribution to design a simulator for time estimation of project management process. Recently, Damodaran et al. (2010) obtained the expected time between failure measures. Further, they showed that the predicted failure times are closer to the actual failure times.

The objective of this work is to make statistical inference from a Bayesian point of view for the Erlang distribution. To be specific, a comparison of assumed informative priors is made through Bayes estimates of the shape and scale parameters of the Erlang distribution. The paper is organized into six sections of which the introduction is the first. Section 2 has brief description of the prior distribution and the loss functions. In Section 3, the posterior means and posterior variances under assumed informative priors are derived. Section 4 is devoted to illustrative example using simulated data set while in Section 5, real data analysis is provided. We conclude with a brief summary of the results in Section 6.

2. Prior and Loss Function

In many practical situations, the information about the shape and scale parameters of the sampling distribution is available in an independent manner. Therefore, here it is assumed that the parameters *c* and *b* are independent a priori and a number of prior distributions have been taken into consideration. These are:

(a) Truncated Poisson distribution as a prior for shape parameter.
(b) Truncated Geometric distribution as a prior for shape parameter.
(c) Inverted Gamma distribution as a prior for scale parameter.
(d) Gamma distribution as a prior for scale parameter.
(e) Truncated Poisson and Inverted Gamma priors for shape and scale parameter.
(f) Truncated Poisson and Gamma priors for shape and scale parameter.
(g) Truncated Geometric and Inverted Gamma priors for shape and scale parameter.
(h) Truncated Geometric and Gamma priors for shape and scale parameter.

Note: Since the shape parameter of Erlang distribution takes values greater than zero (only positive integer), we need such a prior whose value is greater than zero (only positive integer) and thus we have used zero truncated distributions as priors

The loss function considered in this paper is squared error loss function. The squared error loss function for the shape parameter *c* and the scale parameter *b* are defined as

$$L(\hat{c}) = (\hat{c} - c)^2 \tag{2.1}$$

$$L(\hat{b}) = (\hat{b} - b)^2 \tag{2.2}$$

which is symmetric and c, b and $\hat{c}$, $\hat{b}$ represent the true and estimated values of the parameters. This loss function assigns a small weight on estimates near the true value and proportionately more weight on extreme deviation from the true value of the parameter. This loss function is often used because of its analytical tractability in Bayesian analysis.

3.Derivations of Posterior Distributions under Different Informative Priors

The posterior distributions using different informative priors for unknown parameters c (shape) and b (scale) are derived in the following subsequent sub-sections. The probability density function (pdf) of Erlang distribution is given by

$$f(x;c,b) = \frac{x^{c-1}\exp\left(-b^{-1}x\right)}{\Gamma(c)\,b^{c}}, c = 1,2,3,\ldots, b > 0, x > 0.$$

Let $X_1, X_2, \ldots, X_n$ be a random sample from the Erlang distribution, the likelihood function of the sample observations: $\mathbf{x} : x_1, x_2, \ldots, x_n$ is defined as

$$L(c, b; \mathbf{x}) = \frac{\prod_{i=1}^{n} x_i^{c-1} \exp\left(-b^{-1} \sum_{i=1}^{n} x_i\right)}{(\Gamma(c))^n b^{nc}}, c = 1,2,3, \mathbf{h}, b > 0.$$

3.1 When shape parameter *c* is unknown and scale parameter *b* is known

It is well known that, for the Bayes estimators, the performance depends on the form of the prior distribution and the loss function assumed. In this section, we assume two different informative prior distributions for the shape parameter *c*, viz., truncated Poisson distribution and truncated geometric distribution and obtain the Bayes estimators and posterior variances.

3.1.1 Truncated Poisson distribution as a prior for shape parameter *c*

The first prior assumed for shape parameter *c* is truncated Poisson distribution, having following pdf

$$g_1(c;\theta_1) = \frac{\exp(-\theta_1)\theta_1^c}{\Gamma(c+1)(1-\exp(-\theta_1))}, c = 1,2,3, \mathbf{h}, \theta_1 > 0. \qquad (3.1.1.1)$$

Bayesian statistics have the advantage of being able to combine the subjective knowledge of the prior distribution with the knowledge contained in the data. Estimates can be obtained with relatively little data, which become extremely important in the case of expensive testing procedures. By combining the likelihood function (2.2) and the prior density (2.1.1.1), the posterior distribution of *c* given data is

$$g_1(c \mid \mathbf{x}) = \frac{\dfrac{\theta_1^c \exp\left(c \sum_{i=1}^{n} \ln x_i\right)}{\Gamma(c+1)(\Gamma(c))^n b^{nc}}}{\sum_{c=1}^{\infty}\left(\dfrac{\theta_1^c \exp\left(c \sum_{i=1}^{n} \ln x_i\right)}{\Gamma(c+1)(\Gamma(c))^n b^{nc}}\right)}, c = 1, 2, 3, \mathbf{h}. \qquad (3.1.1.2)$$

The Bayes estimator under squared error loss function with the prior $g_1(c;\theta_1)$ is

$$\hat{c}_1 \mid \mathbf{x} = \frac{\sum_{c=1}^{\infty}\left(\frac{c\,\theta_1^{c}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)}{\Gamma(c+1)(\Gamma(c))^{n}\,b^{nc}}\right)}{\sum_{c=1}^{\infty}\left(\frac{\theta_1^{c}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)}{\Gamma(c+1)(\Gamma(c))^{n}\,b^{nc}}\right)} \tag{3.1.1.3}$$

Since, the posterior risk (minimum posterior expected loss) of $\hat{c}_1 \mid \mathbf{x}$ is given by the posterior variance, therefore, the posterior variance of Bayes estimator $\hat{c}_1 \mid \mathbf{x}$ is given by

$$Var\left(\hat{c}_1 \mid \mathbf{x}\right) = \left(\frac{\sum_{c=1}^{\infty}\left(\frac{c^{2}\theta_1^{c}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)}{\Gamma(c+1)(\Gamma(c))^{n}\,b^{nc}}\right)}{\sum_{c=1}^{\infty}\left(\frac{\theta_1^{c}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)}{\Gamma(c+1)(\Gamma(c))^{n}\,b^{nc}}\right)}\right) - \left(\hat{c}_1 \mid \mathbf{x}\right)^{2}. \tag{3.1.1.4}$$

3.1.2 Truncated Geometric distribution as a prior for shape parameter *c*

The second prior assumed for shape parameter c is truncated Geometric distribution, having following pdf

$$g_2\left(c;\theta_2\right) = \theta_2\left(1-\theta_2\right)^{c-1},\ c = 1,2,3,\ldots,\ 0 < \theta_2 < 1. \tag{3.1.2.1}$$

The posterior distribution of c given data under the prior $g_2(c;\theta_2)$ is

$$g_2(c \mid \mathbf{x}) = \frac{\frac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)}{(\Gamma(c))^{n}\,b^{nc}}}{\sum_{c=1}^{\infty}\left(\frac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)}{(\Gamma(c))^{n}\,b^{nc}}\right)},\ c = 1,2,3,\ldots. \tag{3.1.2.2}$$

The Bayes estimator under squared error loss function under the prior $g_2(c;\theta_2)$ is

$$\hat{c}_2 \mid \mathbf{x} = \frac{\sum_{c=1}^{\infty}\left(\frac{c(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)}{(\Gamma(c))^n b^{nc}}\right)}{\sum_{c=1}^{\infty}\left(\frac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)}{(\Gamma(c))^n b^{nc}}\right)}. \tag{3.1.2.3}$$

Similarly, the posterior variance of Bayes estimator $\hat{c}_2 \mid \mathbf{x}$ is given by

$$Var(\hat{c}_2 \mid \mathbf{x}) = \left(\frac{\sum_{c=1}^{\infty}\left(\frac{c^2(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)}{(\Gamma(c))^n b^{nc}}\right)}{\sum_{c=1}^{\infty}\left(\frac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)}{(\Gamma(c))^n b^{nc}}\right)}\right) - (\hat{c}_2 \mid \mathbf{x})^2. \tag{3.1.2.4}$$

3.2 When scale parameter *b* is unknown and shape parameter *c* is known

For the scale parameter *b* of Erlang distribution, the assumed informative priors are Inverted Gamma and Gamma distributions. The Bayes estimators and posterior variances for scale parameter *b* of Erlang distribution are derived in the following sub-sections.

3.2.1 Inverted Gamma distribution as a prior for scale parameter *b*

The first prior assumed for scale parameter *b* is Inverted Gamma distribution, having the following pdf

$$g_1(b;\alpha_1,\beta_1) = \frac{\beta_1^{\alpha_1} b^{-(\alpha_1+1)}\exp\left(-b^{-1}\beta_1\right)}{\Gamma(\alpha_1)}, b > 0, (\alpha_1,\beta_1) > 0. \tag{3.2.1.1}$$

Combining the likelihood function (2.2) and the prior density (2.2.1.1), the posterior distribution of *b* given data is

$$g_1(b\,|\,\mathbf{x})=\frac{\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}\exp\left\{-b^{-1}\left(\beta_1+\sum_{i=1}^{n}x_i\right)\right\}}{\Gamma(\alpha_1+nc)\,b^{+(\alpha_1+nc+1)}},b>0, \tag{3.2.1.2}$$

which is the pdf of Inverted Gamma distribution.

The Bayes estimator under squared error loss function with prior density (2.2.1.1) is

$$\hat{b}_1\,|\,\mathbf{x}=\frac{\beta_1+\sum_{i=1}^{n}x_i}{\alpha_1+nc-1}. \tag{3.2.1.3}$$

The posterior variance of Bayes estimator $\hat{b}_1\,|\,\mathbf{x}$ is given by

$$Var\left(\hat{b}_1\,|\,\mathbf{x}\right)=\frac{\left(\beta_1+\sum_{i=1}^{n}x_i\right)^2}{(\alpha_1+nc-1)^2(\alpha_1+nc-2)}. \tag{3.2.1.4}$$

3.2.2 Gamma distribution as a prior for scale parameter *b*

The second prior assumed for scale parameter *b* is Gamma distribution, having following pdf

$$g_2(b;\alpha_2,\beta_2)=\frac{\beta_2^{\alpha_2}b^{\alpha_2-1}\exp(-\beta_2 b)}{\Gamma(\alpha_2)},b>0,(\alpha_2,\beta_2)>0. \tag{3.2.2.1}$$

Combining the likelihood function (2.2) and the prior density $g_2(b;\alpha_2,\beta_2)$, the posterior distribution of *b* given data is

$$g_2(b\,|\,\mathbf{x})=\frac{\left(\beta_2\left(\sum_{i=1}^{n}x_i\right)^{-1}\right)^{\frac{\alpha_2-nc}{2}}b^{(\alpha_2-nc)-1}\exp\left[-\frac{1}{2}\left\{(2\beta_2)b+\frac{\left(2\sum_{i=1}^{n}x_i\right)}{b}\right\}\right]}{2K_{\alpha_2-nc}\left(2\sqrt{\left(\sum_{i=1}^{n}x_i\right)(\beta_2)}\right)}, \tag{3.2.2.2}$$

which is the pdf of Generalized Inverse Gaussian distribution. Here $K_t(.)$ is a modified Bessel function of third kind with index *t*.

The Bayes estimator under squared error loss function with prior density (2.2.2.1) is

$$\hat{b}_2 \mid \mathbf{x} = \frac{\sqrt{\sum_{i=1}^{n} x_i}\; K_{\alpha_2 - nc + 1}\left(2\sqrt{\left(\sum_{i=1}^{n} x_i\right)(\beta_2)}\right)}{\sqrt{\beta_2}\; K_{\alpha_2 - nc}\left(2\sqrt{\left(\sum_{i=1}^{n} x_i\right)(\beta_2)}\right)}. \tag{3.2.2.3}$$

The posterior variance of Bayes estimator $\hat{b}_2 \mid \mathbf{x}$ is given by

$$Var\left(\hat{b}_2 \mid \mathbf{x}\right) = \frac{\sqrt{\sum_{i=1}^{n} x_i}}{\sqrt{\beta_2}}\left[\left(\frac{K_{\alpha_2 - nc + 2}\left(2\sqrt{\left(\sum_{i=1}^{n} x_i\right)(\beta_2)}\right)}{K_{\alpha_2 - nc}\left(2\sqrt{\left(\sum_{i=1}^{n} x_i\right)(\beta_2)}\right)}\right) - \left(\frac{K_{\alpha_2 - nc + 1}\left(2\sqrt{\left(\sum_{i=1}^{n} x_i\right)(\beta_2)}\right)}{K_{\alpha_2 - nc}\left(2\sqrt{\left(\sum_{i=1}^{n} x_i\right)(\beta_2)}\right)}\right)^2\right]. \tag{3.2.2.4}$$

3.3 When both shape and scale parameters are unknown

In this section, for both unknown parameters of Erlang distribution, the two different independent prior distributions are assumed. The Bayes estimators and posterior variances for both shape parameter *c* and scale parameter *b* of Erlang distribution are derived in the following sub-sections.

3.3.1 Posterior distributions under truncated Poisson and Inverted Gamma priors

Firstly, for the shape parameter *c* of the Erlang distribution, the assumed prior is truncated Poisson distribution, whose pdf is given by

$$g_{11}(c;\theta_1) = \frac{\exp(-\theta_1)\theta_1^{c}}{\Gamma(c+1)(1-\exp(-\theta_1))}, \; c = 1,2,3, \;\; \mathbf{h} \;, \theta_1 > 0. \tag{3.3.1.1}$$

The second informative prior for scale parameter *b* is assumed to be Inverted Gamma distribution, having the following pdf

$$g_{12}(b;\alpha_1,\beta_1) = \frac{\beta_1^{\alpha_1} b^{-(\alpha_1+1)} \exp\left(-b^{-1}\beta_1\right)}{\Gamma(\alpha_1)}, b > 0, \alpha_1 > o, \beta_1 > 0. \tag{3.3.1.2}$$

The joint prior distribution of c and b is defined as

$$g_1(c,b) = g_{11}(c;\theta_1)g_{12}(b;\alpha_1,\beta_1), \tag{3.3.1.3}$$

$$g_1(c,b) \propto \frac{\theta_1^c b^{-(\alpha_1+1)}\exp\left(-b^{-1}\beta_1\right)}{\Gamma(c+1)}, c = 1,2,3, \mathbf{h}, b > 0. \tag{3.3.1.4}$$

Combining the likelihood function (2.2) and the joint prior (2.3.1.4), the joint posterior distribution of c and b given data is

$$g_1(c,b\mid \mathbf{x}) \propto \frac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right) b^{-(\alpha_1+nc+1)}\exp\left\{-b^{-1}\left(\beta_1+\sum_{i=1}^{n}x_i\right)\right\}}{(\Gamma(c))^n\,\Gamma(c+1)}. \tag{3.3.1.5}$$

After simplification, we get the complete joint posterior distribution as

$$g_1(c,b\mid \mathbf{x}) = \frac{\dfrac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right) b^{-(\alpha_1+nc+1)}\exp\left\{-b^{-1}\left(\beta_1+\sum_{i=1}^{n}x_i\right)\right\}}{(\Gamma(c))^n\,\Gamma(c+1)}}{\sum_{c=1}^{\infty}\left(\dfrac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n\,\Gamma(c+1)\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}. \tag{3.3.1.6}$$

The marginal posterior distributions of c and b given data are

$$g_1(c\mid \mathbf{x}) = \frac{\dfrac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{\left\{(\Gamma(c))^n\,\Gamma(c+1)\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}\right\}}}{\sum_{c=1}^{\infty}\left(\dfrac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n\,\Gamma(c+1)\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}, c = 1,2,3, \mathbf{h}, \tag{3.3.1.7}$$

and

$$g_1(b \mid \mathbf{x}) = \frac{\sum_{c=1}^{\infty}\left(\frac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right) b^{-(nc)}}{(\Gamma(c))^n \Gamma(c+1)}\right)\frac{\exp\left\{-b^{-1}\left(\beta_1+\sum_{i=1}^{n}x_i\right)\right\}}{b^{\alpha_1+1}}}{\sum_{c=1}^{\infty}\left(\frac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n \Gamma(c+1)\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}, b > 0.$$

(3.3.1.8)

Under squared error loss function, the Bayes estimators are the means of their posterior distributions. The expressions for Bayes estimators of c and b with their respective posterior variances are given below.

$$\hat{c}_1 \mid \mathbf{x} = \frac{\sum_{c=1}^{\infty}\left(\frac{c\,\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n \Gamma(c+1)\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}{\sum_{c=1}^{\infty}\left(\frac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n \Gamma(c+1)\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)},$$

(3.3.1.9)

$$\hat{b}_1 \mid \mathbf{x} = \frac{\sum_{c=1}^{\infty}\left(\frac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc-1)}{(\Gamma(c))^n \Gamma(c+1)\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc-1}}\right)}{\sum_{c=1}^{\infty}\left(\frac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n \Gamma(c+1)\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)},$$

3.3.1.10)

$$Var\left(\hat{c}_1 \mid \mathbf{x}\right)=\left(\frac{\sum_{c=1}^{\infty}\left(\frac{c^2\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma\left(\alpha_1+nc\right)}{\left(\Gamma(c)\right)^n\Gamma(c+1)\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}{\sum_{c=1}^{\infty}\left(\frac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma\left(\alpha_1+nc\right)}{\left(\Gamma(c)\right)^n\Gamma(c+1)\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}\right)-\left(\hat{c}_1 \mid \mathbf{x}\right)^2$$

(3.3.1.11)

and

$$Var\left(\hat{b}_1 \mid \mathbf{x}\right)=\left(\frac{\sum_{c=1}^{\infty}\left(\frac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma\left(\alpha_1+nc-2\right)}{\left(\Gamma(c)\right)^n\Gamma(c+1)\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc-2}}\right)}{\sum_{c=1}^{\infty}\left(\frac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma\left(\alpha_1+nc\right)}{\left(\Gamma(c)\right)^n\Gamma(c+1)\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}\right)-\left(\hat{b}_1 \mid \mathbf{x}\right)^2.$$

(3.3.1.12)

3.3.2 Posterior distributions under truncated Poisson and Gamma priors

The prior for *c* is again assumed to be truncated Poisson distribution, where as for *b*, the assumed prior is Gamma distribution, having the following pdf

$$g_{11}(c;\theta_1)=\frac{\exp(-\theta_1)\theta_1^c}{\Gamma(c+1)\{1-\exp(-\theta_1)\}},\, c=1,2,3,\ldots,\theta_1>0.$$

(3.3.2.1)

$$g_{21}(b;\alpha_2,\beta_2)=\frac{\beta_2^{\alpha_2}b^{\alpha_2-1}\exp(-b\beta_2)}{\Gamma(\alpha_2)},\, b>0,\alpha_2>o,\beta_2>0.$$

(3.3.2.2)

Now, the joint prior of *c* and *b* is defined as

$$g_2(c,b)=g_{11}(c;\theta_1)g_{21}(b;\alpha_2,\beta_2), \tag{3.3.2.3}$$

$$g_2(c,b)\propto\frac{\theta_1^c b^{\alpha_2-1}\exp(-b\beta_2)}{\Gamma(c+1)},\, c=1,2,3,\ldots,b>0. \tag{3.3.2.4}$$

Combining the likelihood function (2.2) and joint prior density (2.3.2.4), the joint posterior distribution of *c* and *b* given data are given by

$$g_2(c,b\mid \mathbf{x}) \propto \frac{\theta_1^c b^{(\alpha_2-nc)-1}\exp\left\{-\frac{1}{2}\left[(2\beta_2)b+\frac{\left(2\sum_{i=1}^n x_i\right)}{b}\right]\right\}}{\exp\left(-c\sum_{i=1}^n \ln x_i\right)(\Gamma(c))^n\,\Gamma(c+1)}, \tag{3.3.2.5}$$

$$g_2(c,b\mid \mathbf{x}) = \frac{\dfrac{\theta_1^c b^{(\alpha_2-nc)-1}\exp\left\{-\frac{1}{2}\left[(2\beta_2)b+\frac{\left(2\sum_{i=1}^n x_i\right)}{b}\right]\right\}}{\exp\left(-c\sum_{i=1}^n \ln x_i\right)(\Gamma(c))^n\,\Gamma(c+1)}}{\sum_{c=1}^{\infty}\left(\dfrac{2\theta_1^c\exp\left(c\sum_{i=1}^n \ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^n x_i\right)}\right)}{(\Gamma(c))^n\,\Gamma(c+1)\left(\sqrt{\left(\sum_{i=1}^n x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}. \tag{3.3.2.6}$$

The marginal posterior distributions of c and b given data are

$$g_2(c\mid \mathbf{x}) = \frac{\left(\dfrac{2\theta_1^c\exp\left(c\sum_{i=1}^n \ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^n x_i\right)}\right)}{(\Gamma(c))^n\,\Gamma(c+1)\left(\sqrt{\left(\sum_{i=1}^n x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}{\sum_{c=1}^{\infty}\left(\dfrac{2\theta_1^c\exp\left(c\sum_{i=1}^n \ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^n x_i\right)}\right)}{(\Gamma(c))^n\,\Gamma(c+1)\left(\sqrt{\left(\sum_{i=1}^n x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}. \tag{3.3.2.7}$$

and

$$g_2(b \mid \mathbf{x}) = \frac{\sum_{c=1}^{\infty}\left(\frac{\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)\exp\left[-\frac{1}{2}\left\{(2\beta_2)b + b^{-1}\left(2\sum_{i=1}^{n}x_i\right)\right\}\right]}{b^{-(\alpha_2-nc)+1}(\Gamma(c))^n\Gamma(c+1)}\right)}{\sum_{c=1}^{\infty}\left(\frac{2\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\Gamma(c+1)\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}. \tag{3.3.2.8}$$

The expressions for Bayes estimators of c and b with their respective posterior variances are given below.

$$\hat{c}_2 \mid \mathbf{x} = \frac{\sum_{c=1}^{\infty}\left(\frac{2c\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\Gamma(c+1)\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}{\sum_{c=1}^{\infty}\left(\frac{2\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\Gamma(c+1)\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}, \tag{3.3.2.9}$$

$$\hat{b}_2 \mid \mathbf{x} = \frac{\sum_{c=1}^{\infty}\left(\frac{2\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc+1}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\,\Gamma(c+1)\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc+1}}\right)}{\sum_{c=1}^{\infty}\left(\frac{2\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\,\Gamma(c+1)\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}, \tag{3.3.2.10}$$

$$Var(\hat{c}_2 \mid \mathbf{x}) = \left(\frac{\sum_{c=1}^{\infty}\left(\frac{2c^2\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\,\Gamma(c+1)\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}{\sum_{c=1}^{\infty}\left(\frac{2\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\,\Gamma(c+1)\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}\right) - (\hat{c}_2 \mid \mathbf{x})^2 \tag{3.3.2.11}$$

and

$$Var(\hat{b}_2 \mid \mathbf{x}) = \left(\frac{\sum_{c=1}^{\infty}\left(\frac{2\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc+2}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\,\Gamma(c+1)\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc+2}}\right)}{\sum_{c=1}^{\infty}\left(\frac{2\theta_1^c \exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\,\Gamma(c+1)\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}\right) - (\hat{b}_2 \mid \mathbf{x})^2. \tag{3.3.2.12}$$

3.3.3 Posterior distributions under truncated Geometric and Inverted Gamma priors

For the shape parameter c of the Erlang distribution, the assumed prior is truncated Geometric distribution, whose pdf is given by

$$g_{31}(c;\theta_2)=\theta_2(1-\theta_2)^{c-1}, c=1,2,3,\ldots,0<\theta_2<1. \quad (3.3.3.1)$$

The second informative prior for scale parameter b is assumed to be Inverted Gamma distribution, having following pdf

$$g_{12}(b;\alpha_1,\beta_1)=\frac{\beta_1^{\alpha_1} b^{-(\alpha_1+1)}\exp\left(-b^{-1}\beta_1\right)}{\Gamma(\alpha_1)}, b>0, \alpha_1>o, \beta_1>0. \quad (3.3.3.2)$$

The joint prior distribution of c and b is defined as

$$g_3(c,b)\propto (1-\theta_2)^{c-1} b^{-(\alpha_1+1)}\exp\left(-b^{-1}\beta_1\right), c=1,2,3,\ \ldots, b>0. \quad (3.3.3.3)$$

Combining the likelihood function and joint prior density (2.3.3.3), the joint posterior distribution of c and b given data is

$$g_3(c,b|\mathbf{x})\propto\frac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)}{(\Gamma(c))^n b^{+(\alpha_1+nc+1)}\exp\left\{b^{-1}\left(\beta_1+\sum_{i=1}^{n}x_i\right)\right\}}, c=1,2,3,\ldots,b>0.$$

(3.3.3.4)

After simplification, we can write the complete joint posterior distribution as

$$g_3(c,b\mid\mathbf{x})=\frac{\dfrac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)b^{-(\alpha_1+nc+1)}\exp\left\{-b^{-1}\left(\beta_1+\sum_{i=1}^{n}x_i\right)\right\}}{(\Gamma(c))^n}}{\sum_{c=1}^{\infty}\left(\dfrac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}.$$

(3.3.3.5)

The marginal posterior distributions of c and b given data are

$$g_3(c \mid \mathbf{x}) = \frac{\dfrac{(1-\theta_1)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{\left\{(\Gamma(c))^n\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}\right\}}}{\sum_{c=1}^{\infty}\left(\dfrac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}, c = 1,2,3, \mathbf{h} , \tag{3.3.3.6}$$

and

$$g_3(b \mid \mathbf{x}) = \frac{\sum_{c=1}^{\infty}\left(\dfrac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)b^{-(nc)}}{(\Gamma(c))^n}\right)\dfrac{e^{-\frac{\left(\beta_1+\sum_{i=1}^{n}x_i\right)}{b}}}{b^{\alpha_1+1}}}{\sum_{c=1}^{\infty}\left(\dfrac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}, b > 0. \tag{3.3.3.7}$$

The expressions for Bayes estimators of c and b under squared error loss function with their respective posterior variances are given below.

$$\hat{c}_3 \mid \mathbf{x} = \frac{\sum_{c=1}^{\infty}\left(\dfrac{c(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}{\sum_{c=1}^{\infty}\left(\dfrac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}, \tag{3.3.3.8}$$

$$\hat{b}_3 \mid \mathbf{x} = \frac{\sum_{c=1}^{\infty}\left(\frac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc-1)}{(\Gamma(c))^n\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc-1}}\right)}{\sum_{c=1}^{\infty}\left(\frac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}, \tag{3.3.3.9}$$

$$Var(\hat{c}_3 \mid \mathbf{x}) = \left(\frac{\sum_{c=1}^{\infty}\left(\frac{c^2(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}{\sum_{c=1}^{\infty}\left(\frac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}\right) - (\hat{c}_3 \mid \mathbf{x})^2 \tag{3.3.3.10}$$

and

$$Var(\hat{b}_3 \mid \mathbf{x}) = \left(\frac{\sum_{c=1}^{\infty}\left(\frac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc-2)}{(\Gamma(c))^n\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc-2}}\right)}{\sum_{c=1}^{\infty}\left(\frac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)\Gamma(\alpha_1+nc)}{(\Gamma(c))^n\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+nc}}\right)}\right) - (\hat{b}_3 \mid \mathbf{x})^2. \tag{3.3.3.11}$$

3.3.4 Posterior distributions under truncated Geometric and Gamma priors

The prior for c is again assumed to be truncated Geometric distribution, where as for b, the assumed prior is Gamma distribution, having the following pdf

$$g_{31}(c;\theta_2)=\theta_2(1-\theta_2)^{c-1}, c=1,2,3,\ldots, 0<\theta_2<1. \tag{3.3.4.1}$$

$$g_{21}(b;\alpha_2,\beta_2)=\frac{\beta_2^{\alpha_2} b^{\alpha_2-1}\exp(-b\beta_2)}{\Gamma(\alpha_2)}, b>0, \alpha_2>o, \beta_2>0. \tag{3.3.4.2}$$

Now, the joint prior of c and b is defined as

$$g_4(c,b)= g_{31}(c;\theta_2)g_{21}(b;\alpha_2,\beta_2), \tag{3.3.4.3}$$

$$g_4(c,b)\propto (1-\theta_2)^{c-1} b^{\alpha_2-1}\exp(-b\beta_2), c=1,2,3, \ldots, b>0. \tag{3.3.4.4}$$

By combining the likelihood function (2.2) and joint prior density (2.3.4.4), the joint posterior distribution of c and b given data is given by

$$g_4(c,b\mid \mathbf{x})=\frac{\dfrac{(1-\theta_2)^{c-1} b^{(\alpha_2-nc)-1}\exp\left\{-\frac{1}{2}\left((2\beta_2)b+\frac{\left(2\sum_{i=1}^{n}x_i\right)}{b}\right)\right\}}{\exp\left(-c\sum_{i=1}^{n}\ln x_i\right)(\Gamma(c))^n}}{\sum_{c=1}^{\infty}\left(\dfrac{2(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}. \tag{3.3.4.5}$$

The marginal posterior distributions of c and b given data are

$$g_2(c\mid \mathbf{x})=\frac{\left(\dfrac{2(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}{\sum_{c=1}^{\infty}\left(\dfrac{2(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}, c=1,2,3, \ldots \tag{3.3.4.6}$$

and

$$g_2(b|\mathbf{x})=\frac{\sum_{c=1}^{\infty}\left(\frac{(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)\exp\left[-\frac{1}{2}\left\{(2\beta_2)b+b^{-1}\left(2\sum_{i=1}^{n}x_i\right)\right\}\right]}{b^{-(\alpha_2-nc)+1}(\Gamma(c))^n}\right)}{\sum_{c=1}^{\infty}\left(\frac{2(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)},b>0. \tag{3.3.4.7}$$

The expressions for Bayes estimators of c and b with their respective posterior variances are given below.

$$\hat{c}_4\mid\mathbf{x}=\frac{\sum_{c=1}^{\infty}\left(\frac{2c(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}{\sum_{c=1}^{\infty}\left(\frac{2(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}, \tag{3.3.4.8}$$

$$\hat{b}_4 \mid \mathbf{x} = \frac{\sum_{c=1}^{\infty}\left(\dfrac{2(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc+1}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc+1}}\right)}{\sum_{c=1}^{\infty}\left(\dfrac{2(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{(\Gamma(c))^n\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}, \tag{3.3.4.9}$$

$$Var(\hat{c}_4 \mid \mathbf{x}) = \left(\frac{\sum_{c=1}^{\infty}\left(\dfrac{2c^2(1-\theta_2)^{c-1}K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{\exp\left(-c\sum_{i=1}^{n}\ln x_i\right)(\Gamma(c))^n\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}{\sum_{c=1}^{\infty}\left(\dfrac{2(1-\theta_2)^{c-1}K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{\exp\left(-c\sum_{i=1}^{n}\ln x_i\right)(\Gamma(c))^n\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}\right) - (\hat{c}_4 \mid \mathbf{x})^2 \tag{3.3.4.10}$$

and

$$Var\left(\hat{b}_4|\mathbf{x}\right)=\left(\frac{\sum_{c=1}^{\infty}\left(\frac{2(1-\theta_2)^{c-1}K_{\alpha_2-nc+2}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{\exp\left(-c\sum_{i=1}^{n}\ln x_i\right)\left(\Gamma(c)\right)^n\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc+2}}\right)}{\sum_{c=1}^{\infty}\left(\frac{2(1-\theta_2)^{c-1}\exp\left(c\sum_{i=1}^{n}\ln x_i\right)K_{\alpha_2-nc}\left(2\sqrt{\beta_2\left(\sum_{i=1}^{n}x_i\right)}\right)}{\exp\left(-c\sum_{i=1}^{n}\ln x_i\right)\left(\Gamma(c)\right)^n\left(\sqrt{\left(\sum_{i=1}^{n}x_i\right)^{-1}\beta_2}\right)^{\alpha_2-nc}}\right)}\right)-\left(\hat{b}_4|\mathbf{x}\right)^2. \quad (3.3.4.11)$$

4 Simulation Study

In this section, a detailed simulation study is discussed in order to gain insight about the efficiency of Bayes estimators under the assumed informative priors. In each replication, owing to the random sample, the quantities $\sum_{i=1}^{n} x_i$ and $\sum_{i=1}^{n} \ln x_i$ are also random, in order to fix these quantities for a fixed random sample, we performed 10,000 Monte Carlo simulations for these quantities and then Bayes estimates have been calculated.

4.1 Comparison of Bayes estimates when *c* is unknown

For the shape parameter *c* of Erlang distribution, we assumed two discrete prior distributions i.e. truncated Poisson and truncated Geometric distributions. Random samples of sizes n = 30 and 50 are drawn from Erlang distribution with different choices of c i.e. c = 1, 3, 6 and 9, for all of these choices of c; the value of scale parameter b is fixed i.e. b = 6. For each prior nine different sets of hyperparameters have been assumed and then the Bayes estimates have been calculated. The values of hyperparameters of truncated Poisson and truncated Geometric distributions are θ_1 = 1, 2, 3, 4, 5, 6, 7, 8, 9 and θ_2 = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 respectively. It is observed from Tables 1-4 that under both discrete priors, the Bayes estimates of c are closer to the actual values of c. As the sample size increases from n = 30 to 50, the posterior variances of Bayes estimators decreases. The increase in the value of c from c = 1 to c = 9 causes the variances of Bayes estimators to increase and vice versa. There is no general trend has been observed for these two priors, both priors can be used as informative priors for the shape parameter c of Erlang distribution. Also by using some elicitation techniques, the posterior variances of Bayes estimators can be further improved in order to get valid estimates of population parameters.

Table 1: Bayes estimators and their variances under truncated Poisson distribution for *n* = 30

	$c=1$		$c=3$		$c=6$		$c=9$
$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$
1	1.38833×10^{-8}	2.99815	0.00434	5.75537	0.20063	8.40230	0.28209
1	2.77665×10^{-8}	3.00095	0.00405	.87465	0.14524	8.60308	0.29039
1	4.16498×10^{-8}	3.00272	0.00477	5.92849	0.12193	8.71872	0.27857
1	5.55331×10^{-8}	3.00423	0.00575	.96197	0.11180	8.79722	0.26708
1	6.94163×10^{-8}	3.00562	0.00682	5.98642	0.10797	8.85587	0.25876
1	8.32996×10^{-8}	3.00697	0.00794	6.00603	0.10753	8.90253	0.25340
1	9.71829×10^{-8}	3.00828	0.00908	6.02270	0.10905	8.94134	0.25036
1	1.11066×10^{-7}	3.00957	0.01024	6.03743	0.11177	8.97467	0.24903
1	1.24949×10^{-7}	3.01084	0.01140	6.05079	0.11522	9.00398	0.24896

Table 2: Bayes estimators and their variances under truncated Poisson distribution for *n* = 50

	$c=1$		$c=3$		$c=6$		$c=9$
$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$
1	1.37446×10^{-13}	3.00001	0.00006	5.95864	0.04360	8.71296	0.21342
1	2.73781×10^{-13}	3.00006	0.00009	5.98194	0.02581	8.84036	0.15404
1	4.11449×10^{-13}	3.00011	0.00012	5.99128	0.02083	8.89651	0.12415
1	5.49116×10^{-13}	3.00015	0.00016	5.99701	0.01926	8.92980	0.10812
1	6.85896×10^{-13}	3.00019	0.00019	6.00127	0.01908	8.95287	0.09916
1	8.22675×10^{-13}	3.00023	0.00023	6.00479	0.01959	8.97047	0.09422
1	9.59899×10^{-13}	3.00026	0.00027	6.00787	0.20515	8.98479	0.09174
1	1.09734×10^{-12}	3.00030	0.00031	.01069	0.02167	8.99696	0.09087
1	1.23457×10^{-12}	3.00034	0.00034	6.01331	0.02299	9.00767	0.09107

Table 3: Bayes estimators and their variances under truncated Geometric distribution for *n* = 30

$c=1$		$c=3$		$c=6$		$c=9$	
$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$
1	2.49899×10^{-8}	3.00335	0.00563	6.00322	0.00509	8.99061	0.25957
1	2.22132×10^{-8}	3.00271	0.00528	5.98966	0.11534	8.96002	0.26002
1	1.94366×10^{-8}	3.00203	0.00497	5.97416	0.11690	8.92520	0.26168
1	1.66599×10^{-8}	3.00128	0.00472	5.95590	0.12035	8.88462	0.26497
1	1.38833×10^{-8}	3.00044	0.00456	5.93342	0.12665	8.83584	0.27037
1	1.11066×10^{-8}	2.99942	0.00458	5.90401	0.13746	8.77464	0.27827
1	8.32996×10^{-9}	2.99806	0.00493	5.86194	0.15580	8.69310	0.28836
1	5.55331×10^{-9}	2.99584	0.00613	5.79251	0.18742	8.57414	0.29677
1	2.77665×10^{-9}	2.99024	0.01066	5.64360	0.23929	8.37126	0.28218

Table 4: Bayes estimators and their variances under truncated Geometric distribution for *n* = 50

$c=1$		$c=3$		$c=6$		$c=9$	
$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$
1	2.47358×10^{-13}	3.00013	0.00014	6.00457	0.02100	9.00302	0.09532
1	2.19602×10^{-13}	3.00011	0.00013	6.00212	0.02063	8.99179	0.09553
1	1.92291×10^{-13}	3.00009	0.00012	5.99937	0.02054	8.97896	0.09691
1	1.64757×10^{-13}	3.00008	0.00010	5.99619	0.02086	8.96380	0.10001
1	1.37446×10^{-13}	3.00006	0.00009	5.99230	0.02184	8.94507	0.10573
1	1.09690×10^{-13}	3.00004	0.00008	5.98721	0.02396	8.92043	0.11567
1	8.19345×10^{-14}	3.00001	0.00008	5.97974	0.02831	8.88479	0.13291
1	5.52891×10^{-14}	2.99998	0.00008	5.96644	0.03801	8.82480	0.16405
1	2.73115×10^{-14}	2.99991	0.00011	5.93111	0.06684	8.69054	0.22229

4.2 Comparison of Bayes estimates when *b* is unknown

For the scale parameter *b* of Erlang distribution, we have assumed two informative priors viz., Inverted Gamma and Gamma distributions. With different choices of *b* i.e. *b* = 1, 3, 6 and 9 with fixed *c* = 6, random samples of sizes *n* = 30 and 50 are drawn from Erlang distribution. The assumed values of hyperparameters for the two priors are $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1, 2, 3, 4, 5, 6, 7, 8$ and 9. From Tables 5-8, it is observed that with an increase in sample size, the posterior variances of Bayes estimators decreases and vice versa. The increase in value of *b* causes the posterior variances to increase also. The Bayes estimates under Gamma prior are more precise as compared to Bayes estimates under Inverted Gamma prior for each set of hyerparameters for different values population parameter b. For these choices of hyperparameters, the prior Gamma distribution is superior as compared to Inverted Gamma distribution, additionally, by using valid elicitation methods; the efficiency of Bayes estimators under these informative priors can be increased.

Table 5: Bayes estimators and their variances under Inverted Gamma distribution for *n* = 30

	$b=1$		$b=3$		$b=6$		$b=9$
$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
1.00534	0.00564	3.00548	0.05046	0.01493	0.20212	9.01044	0.45356
1.00531	0.00561	2.99440	0.04981	5.98723	0.19914	8.96619	0.44662
1.00528	0.00558	2.98344	0.04917	5.95982	0.19624	8.92242	0.43983
1.00526	0.00555	2.92760	0.04855	5.93272	0.19339	8.87913	0.43318
1.00523	0.00552	2.96188	0.04793	5.90591	0.19060	8.83630	0.42666
1.00520	0.00549	2.95127	0.04733	5.87939	0.18786	8.79395	0.42029
1.00517	0.00546	2.94078	0.04674	5.85316	0.18518	8.75204	0.41404
1.00514	0.00543	2.93040	0.04616	5.82721	0.18256	8.71059	0.40792
1.00512	0.00540	2.92014	0.04560	5.80153	0.17998	8.66957	0.40193

Table 6: Bayes estimators and their variances under Inverted Gamma distribution for *n* = 50

	$b=1$		$b=3$		$b=6$		$b=9$
$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
1.00310	0.00336	3.00629	0.03022	6.00233	0.12049	9.00455	0.27117
1.00309	0.00335	2.99963	0.02999	5.98571	0.11942	8.97796	0.26867
1.00308	0.00334	2.99301	0.02976	5.96920	0.11837	8.95154	0.26621
1.00307	0.00333	2.98643	0.02953	5.95280	0.11733	8.92530	0.26377
1.00306	0.00332	2.97990	0.02930	5.93651	0.11631	8.89923	0.26137
1.00305	0.00330	2.97340	0.02908	5.92032	0.11529	8.87333	0.25900
1.00304	0.00329	2.96696	0.02886	5.90424	0.11429	8.84760	0.25665
1.00303	0.00328	2.96055	0.02864	5.88827	0.11330	8.82204	0.25434
1.00302	0.00327	2.95418	0.02842	5.87240	0.11232	8.79664	0.25205

Table 7: Bayes estimators and their variances under Gamma distribution for *n* =30

	$b=1$		$b=3$		$b=6$		$b=9$
$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
1.00531	0.00042	2.98335	0.00209	5.88153	0.00556	8.68056	0.00962
1.00525	0.00059	2.95179	0.00282	5.73741	0.00710	8.36339	0.01171
1.00519	0.00072	2.92183	0.00330	5.60743	0.00792	8.08919	0.01257
1.00513	0.00082	2.89332	0.00365	5.48920	0.00839	7.84832	0.01289

1.00528	0.00091	2.86614	0.00391	5.38092	0.00866	7.63403	0.01295
1.00502	0.00100	2.84018	0.00422	5.28113	0.00880	7.44141	0.01286
1.00496	0.00107	2.81534	0.00428	5.18871	0.00886	7.26678	0.01269
1.00491	0.00114	2.79154	0.00441	5.10270	0.00887	7.10730	0.01247
1.00486	0.00120	2.76870	0.00451	5.02235	0.00884	6.96077	0.01223

Table 8: Bayes estimators and their variances under Gamma distribution for *n* = 50

	$b=1$		$b=3$		$b=6$		$b=9$
$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
1.00309	0.00019	2.99296	0.00098	5.92122	0.00267	.80089	0.00473
1.00307	0.00027	2.97356	0.00135	5.82999	0.00354	8.59336	0.00607
1.00305	0.00033	2.95477	0.00161	5.74458	0.00408	8.40483	0.00681
1.00303	0.00038	2.93656	0.00181	5.66434	0.00445	8.23231	0.00724
1.00300	0.00043	2.91890	0.00197	5.58871	0.00471	8.07344	0.00749
1.00298	0.00046	2.90175	0.00210	5.51723	0.00490	7.92638	0.00764
1.00296	0.00050	2.88509	0.00221	5.44950	0.00503	7.78959	0.00771
1.00294	0.00053	2.86889	0.00230	5.38516	0.00513	7.66185	0.00773
1.00292	0.00057	2.85314	0.00239	5.32393	0.00520	7.54211	0.00772

4.3 Comparison of Bayes estimates when both *c* and *b* are unknown

For the both parameters *c* and *b* of Erlang distribution, four different joint prior distributions have been assumed i.e. truncated Poisson and Inverted Gamma, truncated Poisson and Gamma, truncated Geometric and Inverted Gamma, and truncated Geometric and Gamma as a joint informative priors. For the first two joint priors, the assumed values of hyperparameters are $\theta_1 = \alpha_1 = \beta_1 = \alpha_2 = \beta_2 =$ 1, 2, 3, 4, 5, 6, 7, 8 and 9, for last two joint priors, the presumed values of hyperparameters are θ_2 = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and $\alpha_1 = \beta_1 = \alpha_2 = \beta_2$ = 1, 2, 3, 4, 5, 6, 7, 8 and 9. Random sample of size n = 30 is drawn from Erlang distribution with different sets of values of c and b that's $c = b = 1$, $c = 3$ and $b = 6$, $c = b = 6$, $c = 9$ and $b = 6$. Due to complexity in computation, we have taken n = 30 only, which can be increased. From Tables 9-12, it is observed that generally for most of cases the joint prior i.e. truncated Poisson and Gamma prior perform better as compared to other joint priors for the choices $c > 1$ and $b > 1$, whereas in some cases, the joint priors i.e. truncated Geometric and Inverted Gamma prior, truncated Geometric and Gamma priors perform better then truncated Poisson and Gamma prior for $\theta_2 > 5$ and $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 > 5$. As mentioned above, the increase in the values of c and b causes the variances of Bayes estimators to increase and vice versa.

Table 9: Bayes estimators and their variances under truncated Poisson and Inverted Gamma joint prior for *n* = 30

$c=1$ and $b=1$				$c=3$ and $b=6$			
$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
0.00087	0.00086	1.03313	0.03703	2.66422	0.37340	.13358	3.51262
1.00131	0.00131	1.03182	0.03583	3.06294	0.43292	6.11743	2.46384
1.00152	0.00151	1.03072	0.03465	3.36366	0.48661	5.50400	1.76221
1.00518	0.00157	1.02975	0.03352	3.62646	0.54529	5.06232	1.36165
1.00155	0.00155	1.02887	0.03244	3.86959	0.59868	4.71225	1.10761
1.00149	0.00148	1.02807	0.03142	4.09847	0.64452	4.42243	0.92346
1.00140	0.00139	1.02732	0.03046	4.31544	0.68515	4.17727	0.78029
1.00130	0.00130	1.02662	0.02955	4.52232	0.72298	3.96676	0.66671
1.00120	0.00120	1.02596	0.02869	4.72071	0.75910	3.78358	0.56708
$c=6$ and $b=6$				$c=9$ and $b=6$			
$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
4.12035	0.79306	9.16727	5.21479	4.95576	1.11536	11.4242	7.71136
4.93385	1.05895	7.58187	3.20599	6.08241	1.54614	9.22347	4.46402
5.59285	1.27704	6.63921	2.24769	7.01022	1.91110	7.94926	3.01282
6.17942	1.47711	5.97326	1.68250	7.84463	2.24270	7.06588	2.19049
6.72161	1.64948	5.46417	1.31233	8.62210	2.55255	6.40048	1.66853
7.23258	1.81611	5.05667	1.05390	9.35975	2.84623	5.87420	1.31343
7.71965	1.97336	4.72032	0.86539	10.0670	3.12688	5.44413	1.06001
8.18731	2.12270	4.43651	0.72332	10.7498	3.39648	5.08431	0.87257
8.63855	2.26513	4.19295	0.61346	11.4119	3.65637	4.77779	0.73001

Table 10: Bayes estimators and their variances under truncated Poisson and Gamma joint prior for *n* = 30

$c=1$ and $b=1$				$c=3$ and $b=6$			
$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
1.00097	0.00097	1.03182	0.03671	3.17437	0.31821	5.60339	1.19812
1.00164	0.00163	1.02934	0.03530	3.81108	0.43465	4.58182	0.66484
1.00208	0.00207	1.02720	0.03398	4.31035	0.47451	4.00376	0.42242
1.00234	0.00234	1.02535	0.03275	4.71621	0.52966	3.63399	0.31093
1.00250	0.00249	1.02372	0.03161	5.07348	0.57235	3.36208	0.24552
1.00256	0.00255	1.02228	0.03054	5.93093	0.60554	3.15245	0.19931
1.00256	0.00255	1.02100	0.02954	5.67743	0.63856	2.98501	0.16729
1.00252	0.00251	1.01985	0.02859	5.94066	0.66965	2.84658	0.14412
1.00244	0.00243	1.01880	0.02772	6.18485	0.69703	2.72942	0.12621

$c = 6$ and $b = 6$				$c = 9$ and $b = 6$			
$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
5.08780	0.78071	7.02093	1.62832	6.37101	1.10318	8.39211	5.05174
6.43904	1.02913	5.47452	0.77523	8.27851	1.50899	6.39020	0.92471
7.47899	1.22317	4.68397	0.48752	9.76462	1.83537	5.39287	0.56470
8.35410	1.38834	4.17821	0.34736	11.0266	2.11913	4.76393	0.39438
9.12190	1.53450	3.81771	0.26588	12.1423	2.37468	4.31993	0.29733
9.81230	1.66670	3.54357	0.21326	13.1523	2.60951	3.98468	0.23558
10.4433	1.78826	3.32586	0.17678	14.0809	2.82816	3.71989	0.19328
11.0267	1.90100	3.14749	0.15017	14.9442	3.03364	3.50389	0.16275
11.5708	2.00642	2.99786	0.13000	15.7534	3.22806	3.32335	0.13980

Table 11: Bayes estimators and their variances under truncated Geometric and Inverted Gamma joint prior for *n* = 30

$c = 1$ and $b = 1$				$c = 3$ and $b = 6$			
$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
1.00156	0.00155	1.03277	0.03720	.18297	0.56390	.00071	2.76411
1.00105	0.00104	1.03196	0.03577	3.25892	0.55729	5.78613	2.38596
1.00071	0.00070	1.03112	0.03447	3.19710	0.54772	5.61315	2.08476
1.00047	0.00047	1.03029	0.03328	3.36382	0.53508	5.47848	1.84880
1.00031	0.00031	1.02947	0.03218	3.38871	0.51856	5.38122	1.66784
1.00020	0.00019	1.02868	0.03116	3.38992	0.49673	5.32409	1.53454
1.00012	0.00012	1.02792	0.03021	3.35921	0.46717	5.31660	1.44709
1.00007	0.00006	1.02719	0.02933	3.27916	0.42561	5.38475	1.41790
1.00003	0.00002	.02649	0.02848	3.09915	0.36573	5.62121	1.52871
$c = 6$ and $b = 6$				$c = 9$ and $b = 6$			
$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
6.25031	2.25287	6.11105	2.67430	9.22203	4.98138	6.22178	2.73635
6.27886	2.15157	6.03326	2.43091	9.09039	4.57271	6.26844	2.59428
6.27080	2.03742	5.99404	2.24845	8.90790	4.16194	6.35464	2.50154
6.22231	1.90990	5.99536	2.11683	8.66996	3.74793	6.48676	2.45553
6.12720	1.76774	6.04311	2.03150	8.36860	3.32829	6.67653	2.45929
5.97503	1.60819	6.14985	1.99407	7.99006	2.89870	6.94534	2.52423
5.74697	1.42624	6.34179	2.01618	7.50922	2.45135	7.33518	2.67869
5.40384	1.21095	6.68100	2.13441	6.87380	1.97024	7.94288	2.99736
4.83556	0.93227	7.37071	2.48915	5.93792	1.41206	9.08177	3.75328

Table 12: Bayes estimators and their variances under truncated Geometric and Gamma joint prior for *n* = 30

$c=1$ and $b=1$				$c=3$ and $b=6$			
$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
1.00175	0.00174	1.03141	0.03690	3.71645	0.55113	.88198	1.05113
1.00131	0.00130	1.02950	0.03522	4.10361	0.56173	4.30449	0.64841
1.00097	0.00096	1.02777	0.03373	4.38283	0.55255	3.95541	0.45036
1.00071	0.00070	1.02617	0.03239	4.58793	0.54746	3.72416	0.34455
1.00050	0.00050	1.02471	0.03117	4.73911	0.54104	3.56062	0.28223
1.00034	0.00034	1.02337	0.03006	4.84213	0.52741	3.44365	0.24032
1.00022	0.00022	1.02214	0.02904	4.89288	0.50502	3.36580	0.21009
1.00013	0.00012	1.02099	0.02810	4.87419	0.47244	3.32974	0.18835
1.00005	0.00005	1.01994	0.02723	4.72607	0.42223	3.36199	0.17442
$c=6$ and $b=6$				$c=9$ and $b=6$			
$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$	$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
7.39348	2.20673	4.98293	1.08661	10.9797	4.86267	.06232	1.10700
8.09930	2.12302	4.45975	0.66292	11.8982	4.50056	4.59147	0.68771
8.57075	2.03084	4.15798	0.47678	12.4605	4.15117	4.33197	0.50147
8.88633	1.92536	3.96636	0.37331	12.7869	3.79991	4.17933	0.39737
9.07761	1.80544	3.84378	0.30833	12.9263	3.44038	4.09560	0.31901
9.15405	1.66916	3.77323	0.26459	12.8936	3.06839	4.06667	0.28796
9.10659	1.51211	3.75076	0.23417	12.6759	2.67560	4.01988	0.25779
8.89612	1.32401	3.78674	0.21345	12.2152	2.24447	4.18832	0.23802
8.83939	1.07491	3.93119	0.20238	11.3153	1.72605	4.42542	0.22945

5 Real data analysis

This section presents numerical example for the proposed estimates based on a real data set. For illustration of our proposed estimates, we consider the survival time (in weeks) for 20 male rats (see, Lawless (2003)) that were exposed to a high level of radiation. To check the validity of the model, we computed the goodness of fit tests for data set 152, 152, 115, 109, 137, 88, 94, 77, 160, 165, 125, 40, 128, 123, 136, 101, 62, 153, 83 and 69.

Below is the summary of the three goodness of fit tests for this data set:

1. Kolmogrov-Smirnov test: Test statistic: 0.1479 with p-value 0.72019.
2. Anderson-Darling test: Test statistic: 0.49212.
3. Chi-square test: Test statistic: 0.78315 with p-value 0.67599.

From all tests, it is evident that the Erlang distribution with parameters $c=10$ and $b=11.29$ fits the data set well. The results from this data analysis echo the same pattern as found in the simulation study. Complete results are too extensive to be published, so, for illustration, we have computed Bayes estimates under joint priors only and are presented in Tables 13-16. Since we do not have prior information, hence, we have selected several sets of hyperparameters (for the first two joint priors, assumed values of hyperparameters are $\theta_1=\alpha_1=\beta_1=\alpha_2=\beta_2=1,2,3,4,5,6,7,8$ and for last two joint priors, the presumed values of hyperparameters are θ_2 =0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8 and $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1, 2, 3, 4, 5, 6, 7, 8$).

Table 13: Bayes estimators and their variances under truncated Poisson and Inverted Gamma prior

$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
4.30234	1.14695	28.1488	70.2236
5.54421	1.69339	21.4758	34.1738
6.62293	2.18403	17.7676	20.3376
7.62801	2.64809	15.2897	13.3773
8.58958	3.09538	13.4827	9.37685
9.52145	3.53040	12.0935	6.87628
10.4310	3.95556	10.9864	5.21807
11.3228	4.37231	10.0804	4.06832

Table 14: Bayes estimators and their variances under truncated Poisson and Gamma prior

$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
7.33273	1.30198	14.4951	4.903290
10.1243	1.91150	10.4285	1.896850
12.3380	2.41887	8.54560	1.073190
14.2422	2.87021	7.40323	0.713270
15.9437	3.28411	6.61681	0.518609
17.4956	3.70843	6.03361	0.399367
18.8223	8.43320	5.57956	0.316771
20.2913	4.35346	5.21429	0.230579

Table 15: Bayes estimators and their variances under truncated Geometric and Inverted Gamma prior

$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
9.19475	7.10892	13.4732	19.6530
9.07416	6.39122	13.4758	17.7327
8.87917	5.68591	13.6073	16.4715
8.60899	4.99453	13.8746	15.7302
8.25859	4.31605	14.3007	15.4571
7.81651	3.64677	19.9343	15.6860
7.25952	2.97895	15.8770	16.5848
6.53707	2.29640	17.3667	18.6553

Table 16: Bayes estimators and their variances under truncated Geometric and Gamma prior

$\hat{c}_1 \mid \mathbf{x}$	$Var(\hat{c}_1 \mid \mathbf{x})$	$\hat{b}_1 \mid \mathbf{x}$	$Var(\hat{b}_1 \mid \mathbf{x})$
14.0392	7.82008	8.19189	2.69972
16.1623	7.57924	6.95932	1.36773
17.4017	7.14282	6.36972	0.901136
18.1168	6.46527	6.04161	0.676443
18.4556	5.27412	5.86127	0.542894
18.4425	5.19447	5.78542	0.456274
18.1217	4.18651	5.80462	0.413280
17.3399	3.81790	5.93399	0.363667

It is evident from the above Tables 13-16 that the Bayes estimates under truncated Poisson and Gamma prior are more precise as compared to other competitors. However, by using suitable prior information, these Bayes estimates can further be improved.

6. Conclusion

In this study we have assumed several informative priors for shape and scale parameters of the single and both unknown parameters of Erlang distribution. Bayes estimators of unknown parameters are obtained under squared error loss functions. Generally Bayes estimates under each prior are equally efficient but for scale parameter *b*, the Bayes estimates under Gamma prior are more precise as compared to other priors. In case of joint priors, for most of the cases, the truncated Poisson and Gamma prior gave efficient Bayes estimates for both parameters *c* and *b*. The Bayes estimates can be further improved by using some prior elicitation and empirical Bayes methods.

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