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# SUB ADDITIVE MEASURES OF FUZZY INFORMATION

**D.S.Hooda[1] and Divya Jain[2]**
Department of Mathematics
Jaypee Institute of Engineering and Technology
A.B. Road, Raghogarh, Guna
E Mail: 1. ds_hooda@rediffmail.com, 2. agrawal_divya2002@yahoo.co.in

## Abstract

In the present communication, we review the existing measures of fuzzy information. We define and characterize two fuzzy information measures which are sub additive and different from known measures of fuzzy information. We also study monotonic behavior and particular cases of these fuzzy information measures.

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## 1. Introduction

Fuzzy entropy measures the degree of fuzziness of a fuzzy set. It is peculiar to mathematics, information theory, and computer science. It is an important concept in fuzzy set theory and has been successfully applied to pattern recognition, image processing, classifier design and neural network structure, etc. Wang adopted fuzzy entropy as a fast processing for segmentation results as obtained on using cost function. It utilizes the information of regions and achieves better segmentation results than as if we use only cost function.

The concept of entropy was developed by Shannon (1948) to measure the uncertainty of a probability distribution. The concept of fuzzy set was introduced by Zadeh (1965) who also developed his own theory to measure the ambiguity of a fuzzy set.

A fuzzy set is a class of objects with a continuum of grades of membership; such a set is characterized by a membership function which assigns to each object a grade of membership ranging between 0 and 1.

More often the classes of objects encountered in the real physical world do not have precisely defined criteria of membership e.g. the class of animals clearly includes dogs, horses, birds, etc. as its members and clearly excludes such objects as rocks, fluids, plants, etc. However, such objects as starfish, bacteria, etc. have an ambiguous status with respect to the class of animals. Yet, the fact remains that such imprecisely defined classes play an important role in human thinking, particularly in the domains of pattern recognition, communication of information and abstraction.

Let $X = \{x_1, x_2, ..., x_n\}$ be a universe of discourse. A fuzzy set A in X is characterized by a membership function $\mu_A(x_i)$ which associates each point in X with a real number in the interval [0, 1]. In fact $\mu_A(x_i)$ associates with each $x_i \in X$ with a grade of membership on the set *A*. Thus

$$A = \left\{ x_i : \mu_A\left(x_i\right) \in [0,1], \forall x_i \in X \right\} \tag{1.1}$$

where $\mu_A(x_i)$ is a membership function defined as follows:

$$\mu_A\left(x_i\right) = \begin{cases} 0 & \text{if } x_i \text{ does not belongs to } A \text{ and has no ambiguity} \\ 1 & \text{if } x_i \text{ does not belongs to } A \text{ and has no ambiguity} \\ 0.5 & \text{if } x_i \text{ belongs to } A \text{ and has maximum ambiguity} \end{cases}$$

Let $x_1, x_2, \ldots, x_n$ be the members of the universe of discourse, then all $\mu_A(x_1), \mu_A(x_2), \ldots \mu_A(x_n)$ lie between 0 and 1, but these are not probabilities because their sum is not unity. However,

$$\Phi_A(x_i) = \frac{\mu_A(x_i)}{\sum_{i=1}^{n} \mu_A(x_i)}, \qquad i = 1,2,\ldots,n. \tag{1.2}$$

is a probability distribution. On considering (1.2) Kaufman (1980) defined entropy of a fuzzy set *A* having *n* support points by

$$H(A) = -\frac{1}{\log n} \sum_{i=1}^{n} \Phi_A(x_i) \log \Phi_A(x_i). \tag{1.3}$$

In Fuzzy set theory, the entropy is a measure of fuzziness which means the amount of average ambiguity/difficulty in making a decision whether an element belongs to a set or not. A measure of fuzziness H (A) of a fuzzy set A should satisfy at least the following properties:

(P-1) $H(A)$ is minimum value if and only if *A* is a crisp set, i.e. $\mu_A(x_i) = 0$ or 1 for all $x_i$ : $i = 1, 2, \ldots n$.

(P-2) $H(A)$ is maximum value if and only if *A* is most fuzzy set, i.e. $\mu_A(x_i) = 0.5$ for all $x_i$ : $i = 1, 2, \ldots n$ .

(P-3) $H(A) \geq H(A^*)$, where $A^*$ is sharpened version of *A*.

(P-4) $H(A) = H(\bar{A})$, where $\bar{A}$ is the complement of *A*.

Since $\mu_A(x_i)$ and 1 – $\mu_A(x_i)$ for all i = 1,2,…n, gives the same degree of fuzziness, therefore corresponding to Shannon's (1948) entropy, Deluca. and Termini (1971) defined the following measure of fuzzy entropy:

$$H(A) = -\sum_{i=1}^{n} \left[\mu_A(x_i) \log \mu_A(x_i) + \left(1 - \mu_A(x_i)\right) \log\left(1 - \mu_A(x_i)\right)\right]. \tag{1.4}$$

It may be seen that (1.4) satisfies all four properties (P-1) to (P-4) and hence it is a valid measure of fuzzy entropy.

Later on corresponding to Renyi's (1961) entropy, Bhandari and Pal (1993) suggested the following measure of fuzzy entropy:

$$\frac{1}{1-\alpha}\sum_{i=1}^{n}\log\left[\mu_A^{\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha}\right], \alpha\neq 1,\quad \alpha>0 \tag{1.5}$$

Corresponding to Pal and Pal (1989) exponential entropy, Bhandari and Pal (1993) introduced the fuzzy entropy as given below:

$$\frac{1}{n\sqrt{e}-1}\sum_{i=1}^{n}\log\left[\mu_A(x_i)e^{1-\mu_A(x_i)}+\left(1-\mu_A(x_i)\right)e^{\mu_A(x_i)}-1\right] \tag{1.6}$$

Harvda and Charvat (1967) characterized entropy of a discrete probability distribution given by

$$H^{\beta}(P)=\frac{1}{2^{1-\beta}-1}\left[\left(\sum_{i=1}^{n}p_i^{\beta}\right)-1\right]\quad \beta>0,\quad \beta\neq 1 \tag{1.7}$$

Corresponding to Harvda and Charvat (1967) entropy Kapur (1986) has proposed the following measure of fuzzy entropy:

$$H^{\alpha}(A)=(1-\alpha)^{-1}\sum_{i=1}^{n}\left[\mu_A^{\alpha}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha}-1\right]\quad \alpha>0,\ \alpha\neq 1 \tag{1.8}$$

Sharma and Mittal (1975) characterized non-additive entropy of a discrete probability distribution given by

$$H_{\alpha}^{\beta}(P)=\frac{1}{2^{1-\beta}-1}\left[\left(\sum_{i=1}^{n}p_i^{\alpha}\right)^{\frac{\beta-1}{\alpha-1}}-1\right], \alpha\neq\beta,\ \alpha>0, \beta>0, \alpha\neq 1. \tag{1.9}$$

Hooda (2004) suggested the following measure of fuzzy entropy corresponding to Sharma and Mittal's (1975) non-additive entropy of a discrete probability distribution:

$$H_{\alpha}^{\beta}(A)=\frac{1}{2^{1-\beta}-1}\sum_{i=1}^{n}\left[\left(\mu_A^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}\right)^{\frac{\beta-1}{\alpha-1}}-1\right], \tag{1.10}$$

where $\alpha\neq\beta,\ \alpha>0,\ \beta>0, \alpha\neq 1.$

Sharma and Taneja (1977) characterized the following entropies of complete probability distribution:

$$H^{\alpha}(P_i)=2^{\alpha-1}\sum_{i=1}^{n}\left[p_i^{\alpha}(x_i)\log p_i\right],\ \alpha>0 \tag{1.11}$$

$$H_{\beta}^{\alpha}(P_i)=\left(2^{1-\alpha}-2^{1-\beta}\right)^{-1}\sum_{i=1}^{n}\left[p_i^{\alpha}(x_i)-p_i^{\beta}(x_i)\right], \alpha\neq\beta,\quad \alpha>0,\quad \beta>0 \tag{1.12}$$

$$H^{\alpha}(P_i) = \frac{-2^{\alpha-1}}{\sin\beta}\sum_{i=1}^{n}\left[p_i^{\alpha}(x_i)\sin\left(\beta\log p_i\right)\right], \beta \neq k\pi, k = 0,1,2.... \tag{1.13}$$

These entropies are called sub-additive entropies as these are neither additive nor non-additive.

In this paper we define fuzzy entropy of type $\alpha$ corresponding to (1.11) in section 2 and verify its validity. In section 3 we define fuzzy entropy of type $(\alpha, \beta)$ corresponding to (1.12) and verify its validity. We study their monotonic behavior in section 4 and 5 respectively.

## 2. Sub Additive Fuzzy Information Measure of type $\alpha$

Corresponding to sub-additive entropy (1.11) of a complete probability distribution we define the following sub additive fuzzy entropy:

$$H_{\alpha}(A) = -2^{\alpha-1}\sum_{i-1}^{n}\mu_A^{\alpha}(x_i)\log\left(\mu_A(x_i)\right) + \left(1-\mu_A(x_i)\right)^{\alpha}\log\left(\left(1-\mu_A(x_i)\right)\right) \tag{2.1}$$

where $0.5 < \alpha < 2$ *and* $0\log 0 = 0.$

In information theory, entropy symbol *H* (or information entropy or Shannon entropy) is an approximate measure of information measured in units of bits in an electrical signal or message. It is worth mentioning that fuzzy entropy and fuzzy information measure are synonymous. Hence we shall use fuzzy information measure instead of fuzzy entropy in our research work.

**Theorem 1.** The fuzzy information measure given by (2.1) is a valid measure.

**Proof.** To prove that the given measure is a valid measure of fuzzy information, we shall show that (2.1) satisfies the four properties (P-1) to (P-4).

(P-1): $H_{\alpha}(A) = 0$ if only if A is non fuzzy set or crisp set.

We know that

$\mu_A^{\alpha}(x_i)\log\mu_A(x_i) = 0$ *and* $\left(1-\mu_A(x_i)\right)^{\alpha}\log\left(1-\mu_A(x_i)\right) = 0$ if only if $\mu_A(x_i) = 0$ or $\mu_A(x_i) = 1$ : $i$ = 1,2,…$n$, which implies $H_{\alpha}(A) = 0$ if only if A is non fuzzy or crisp set.

(P-2): $H_{\alpha}(A)$ is maximum if and only if A is the most fuzzy set, i.e. $\mu_A(x_i) = 0.5$ for all $i = 1, 2, ...., n$ .

Differentiating $H_{\alpha}(A)$ with respect to $\mu_A(x_i)$, we have

$$\frac{dH_{\alpha}(A)}{d\mu_A(x_i)} = -2^{\alpha-1}\sum_{i=1}^{n}\left[\begin{array}{l}\alpha\mu_A^{\alpha-1}(x_i)\log\mu_A(x_i) + \mu_A^{\alpha-1}(x_i) - \alpha\left(1-\mu_A(x_i)\right)^{\alpha-1} \\ \log\left(1-\mu_A(x_i)\right) - \left(1-\mu_A(x_i)\right)^{\alpha-1}\end{array}\right] \tag{2.2}$$

which vanishes at $\mu_A(x_i)$ =0.5.

Again differentiating $H_\alpha(A)$ with respect to $\mu_A(x_i)$ , we have

$$\frac{d^2 H_\alpha(A)}{d\mu_A(x_i)^2} = -2^{\alpha-1} \sum_{i-1}^{n} \left[ \begin{array}{l} \alpha(\alpha-1)\mu_A^{\alpha-2}(x_i)\log\mu_A(x_i) + \alpha\mu_A^{\alpha-2}(x_i) + \\ (\alpha-1)\mu_A^{\alpha-2}(x_i) + \alpha(\alpha-1)(1-\mu_A(x_i))^{\alpha-2}\log(1-\mu_A(x_i)) \\ +\alpha(1-\mu_A(x_i))^{\alpha-2} + (\alpha-1)(1-\mu_A(x_i))^{\alpha-2} \end{array} \right] \tag{2.3}$$

Putting $\mu_A(x_i) = 0.5$ in (2.3), we have

$$\frac{d^2 H_\alpha(A)}{d\mu_A(x_i)^2} = 4\sum_{i-1}^{n}\left[\alpha^2 - 3\alpha + 1\right].$$

It implies that

$$\frac{d^2 H_\alpha(A)}{d\mu_A(x_i)2} = < 0 \text{ if } \quad 0.5 < \alpha \; < \; 2.$$

Hence $H_\alpha(A)$ is maximum if and only if A is most fuzzy set i.e. $\mu_A(x_i) = 0.5$ : $i$ = 1,2, ..,$n$ and $0.5 < \alpha < 2$.

(P-3): Sharpening reduces the value of Information measure. Let us consider

$$\frac{dH_\alpha(A)}{d\mu_A(x_i)} = -2^{\alpha-1} N \tag{2.4}$$

where

$$N = \sum_{i-1}^{n}\left[\mu_A^{\alpha-1}(x_i)\left[\log\left(\mu_A(x_i)\right)^\alpha + 1\right] - \left(1-\mu_A(x_i)\right)^{\alpha-1}\left[\log\left(1-\mu_A(x_i)\right)^\alpha - 1\right]\right] \tag{2.5}$$

Substituting $y = \mu_A(x_i)$ in (2.5), we have

$$N = \sum_{i=1}^{n} y^{\alpha-1}\log y^\alpha - (1-y)^{\alpha-1}\log(1-y)^\alpha + y^{\alpha-1} - (1-y)^{\alpha-1}.$$

It implies

$$N = \sum_{i=1}^{n} \frac{-1}{y}\left(y^\alpha - y^\alpha \log y^\alpha\right) - \frac{-1}{1-y}\left((1-y)^\alpha - (1-y)^\alpha \log(1-y)^\alpha\right) + 2y^{\alpha-1} - 2(1-y)^{\alpha-1}. \tag{2.6}$$

Using $u - u\log u \le 1$ in (2.6) we get

$$N \le \sum_{i=1}^{n} \frac{-1}{y} + \frac{1}{1-y} + y^{\alpha-1} - (1-y)^{\alpha-1} \le \sum_{i=1}^{n} \frac{-1+2y}{y(1-y)} - 2\frac{y\left(y^{\alpha} + (1-y)^{\alpha}\right) + y^{\alpha}}{y(1-y)} < 0 \tag{2.7}$$

Hence (2.7) gives N <0, where $0 < y < 0.5$. Thus

$$\frac{dH_\alpha(A)}{d\mu_A(x_i)} = -2^{\alpha-1} N > 0 \tag{2.8}$$

Hence $H_\alpha(A)$ is an increasing function of $\mu_A(x_i)$ in the region $0 < \mu_A(x_i) < 0.5$. Similarly, we can prove that $H_\alpha(A)$ is a decreasing function of $\mu_A(x_i)$ in the region $0.5 \le \mu_A(x_i) \le 1$. Hence we can conclude that $H_\alpha(A)$ is a concave function.

Next we prove that sharpening reduces the value of information measure.

Let A* be sharpened version of A which means that

if $\mu_A(x_i) < 0.5$, then $\mu_{A^*}(x_i) \le \mu_A(x_i)$ for all *i*= 1,2, ….*n* and

if $\mu_A(x_i) > 0.5$, then $\mu_{A^*}(x_i) \ge \mu_A(x_i)$ for all *i* = 1,2, ….*n*.

Since $H_\alpha(A)$ is increasing function of $\mu_A(x_i)$ in the region $0 < \mu_A(x_i) < 0.5$ and decreasing function of $\mu_A(x_i)$ in the region $0.5 \le \mu_A(x_i) \le 1$, therefore

(i) $\mu_{A^*}(x_i) \le \mu_A(x_i) \Rightarrow H_\alpha(A^*) \le H_\alpha(A)$ in [0,0.5) (2.9)

(ii) $\mu_{A^*}(x_i) \le \mu_A(x_i) \Rightarrow H_\alpha(A^*) \le H_\alpha(A)$ in [0.5, 1) (2.10)

Hence (2.9) and (2.10) together give $H_\alpha(A^*) \le H_\alpha(A)$.

(P-4): It is evident from the definition that $H_\alpha(A) = H_\alpha(\overline{A})$.

Hence $H_\alpha(A)$ satisfies all the essential four properties of fuzzy information measure. Thus it is a valid measure of sub additive fuzzy information measure. It may be noted that if $\alpha = 1$, (2.1) reduces to (1.4).

## 3. Sub Additive Fuzzy Information Measure of Type $(\alpha, \beta)$

Corresponding to Sharma and Taneja's (1977) entropy (1.12) of a complete probability distribution we define the following sub additive fuzzy information measure of type $(\alpha, \beta)$:

$$H_\alpha^\beta(A) = \frac{1}{\beta-\alpha}\left[\sum_{i-1}^{n} \mu_A^\alpha(x_i) + \left(1-\mu_A(x_i)\right)^\alpha - \mu_A^\beta(x_i) - \left(1-_A(x_i)\right)^\beta\right], \tag{3.1}$$

where $0<\alpha<1$ and $\beta \ge 1$ or $0<\beta<1$ and $\alpha \ge 1$.

**Theorem 2**: Fuzzy information measure given by (3.1) is a valid sub additive measure of type $(\alpha,\beta)$.

**Proof**: The proposed measure $H_\alpha^\beta(A)$ will be valid if and only if it satisfies all the four postulates (P-1) to (P- 4). So we prove these postulates on by one as follows:

To prove this , we consider

$$H_\alpha^\beta(A)=F(A)+G(A), \tag{3.2}$$

where

$$F(A)=\frac{1}{\beta-\alpha}\sum_{i=1}^{n}\mu_A^\alpha(x_i)+\left(1-\mu_A(x_i)\right)^\alpha-1 \tag{3.3}$$

and

$$G(A)=\frac{1}{\alpha-\beta}\sum_{i=1}^{n}\mu_A^\beta(x_i)+\left(1-\mu_A(x_i)\right)^\beta-1 \tag{3.4}$$

To show that

$H_\alpha^\beta(A)$ is a valid fuzzy information measure, it is sufficient to show that (3.3) satisfies the four postulates (P-1) to (P- 4).

(P-1): $F(A)$ =0 if and only if A is a crisp set, i.e. $\mu_A(x_i)$ =0 or 1 for all $i$ = 1, 2…,$n$.

For $\forall\mu_A(x_i)$ and $\forall\alpha>0$, we have $\mu_A^\alpha(x_i)+\left(1-\mu_A(x_i)\right)^\alpha>1$ and $\mu_A^\alpha(x_i)+\left(1-\mu_A(x_i)\right)^\alpha=1$ if and only if A is non-fuzzy set i.e. if $\mu_A(x_i)$ = 0 or 1 for all i = 1,2,…,n. Hence $F(A)$ = 0 if and only if $A$ is non fuzzy set i.e. at $\mu_A(x_i)$ = 0 or 1.

(P-2): $F(A)$ is maximum if and only if A is most fuzzy set i.e. $\mu_A(x_i)$ = 0.5: $i$ = 1, 2…,$n$.

Differentiating $F(A)$ with respect to $\mu_A(x_i)$, we have

$$\frac{dF(A)}{d\mu_A(x_i)}=\frac{1}{\beta-\alpha}\sum_{i=1}^{n}\alpha\mu_A^{\alpha-1}(x_i)-\alpha\left(1-\mu_A(x_i)\right)^{\alpha-1}, \tag{3.5}$$

which vanishes at $\mu_A(x_i)$ = 0.5.

Again differentiating (3.5) with respect to $\mu_A(x_i)$. We have

$$\frac{d^2F(A)}{d\mu_A(x_i)^2}=\frac{1}{\beta-\alpha}\sum_{i=1}^{n}\alpha(\alpha-1)\mu_A^{\alpha-2}(x_i)+\alpha(\alpha-1)\left(1-\mu_A(x_i)\right)^{\alpha-2}$$

or

$$\frac{d^2F(A)}{d\mu_A(x_i)^2}=\frac{1}{\beta-\alpha}\sum_{i=1}^{n}\alpha(\alpha-1)\left[\mu_A^{\alpha-2}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha-2}\right]=\frac{1}{\beta-\alpha}\alpha(\alpha-1)S$$

where

$$\sum_{i=1}^{n}\left[\mu_A^{\alpha-2}(x_i)+\left(1-\mu_A(x_i)\right)^{\alpha-2}\right]=S>0,\quad \forall\ \mu_A(x_i)\ \ and\ \ \forall\alpha>0.$$

Two cases arise:

Case1. When $0<\alpha<1$ and $\beta>1$, we have $\frac{1}{\beta-\alpha}>0$ *and* $\alpha(\alpha-1)<0$.

Hence $\frac{d^2F(A)}{d\mu_A(x_i)^2}<0$.

Case 2. When $0<\beta<1$ and $\alpha>1$, we have $\frac{1}{\beta-\alpha}<0$ *and* $\alpha(\alpha-1)>0$.

Hence $\frac{d^2F(A)}{d\mu_A(x_i)^2}<0$.

Thus in both the cases, (3.3) is maximum if and only if A is most fuzzy set i.e. at $\mu_A(x_i)=0.5$ for all $i=1,2,\ldots,n$.

(P-3): Sharpening reduces the value of Information measure.

From (3.5), we have

$$\frac{dF(A)}{d\mu_A(x_i)}=\frac{1}{\beta-\alpha}\sum_{i=1}^{n}\alpha\mu_A^{\alpha-1}(x_i)-\alpha\left(1-\mu_A(x_i)\right)^{\alpha-1}.$$

Let $0<\mu_A(x_i)<0.5$, then two cases arise:

Case 1.when $0<\alpha<1$ and $\beta>1$, we have

$$\frac{1}{\beta-\alpha}>0\ and\ \alpha\mu_A^{\alpha-1}(x_i)-\alpha\left(1-\mu_A(x_i)\right)^{\alpha-1}=\alpha\left(\mu_A^{\alpha-1}(x_i)-\left(1-\mu_A(x_i)\right)^{\alpha-1}\right)>0$$

It implies $\frac{dF(A)}{d\mu_A(x)}>0$.

Case 2. When $0<\beta<1$ and $\alpha>1$, we have

$$\frac{1}{\beta-\alpha}<0\ and\ \alpha\mu_A^{\alpha-1}(x_i)-\alpha\left(1-\mu_A(x_i)\right)^{\alpha-1}=\alpha\left(\mu_A^{\alpha-1}(x_i)-\left(1-\mu_A(x_i)\right)^{\alpha-1}\right)<0$$

.

It implies $\frac{dF(A)}{d\mu_A(x)}>0$.

Hence in both the cases $F(A)$ is an increasing function of $\mu_A(x_i)$ in the region [0, 0.5). Similarly we can prove that $F(A)$ is a decreasing function of $\mu_A(x_i)$ in the region $0.5 \le \mu_A(x_i) \le 1$. Hence $F(A)$ is a concave function.

Let A* be sharpened version of A, i.e. if $\mu_A(x_i) < 0.5$, then $\mu_{A^*}(x_i) \le \mu_A(x_i)$ and if $\mu_A(x_i) > 0.5$, then $\mu_{A^*}(x_i) \ge \mu_A(x_i)$.

Since $F(A)$ is increasing function of $\mu_A(x_i)$ in the region $0 < \mu_A(x_i) < 0.5$ and decreasing function of $\mu_A(x_i)$ in the region $0.5 \le \mu_A(x_i) \le 1$, therefore

$$\mu_{A^*}(x_i) \le \mu_A(x_i) \Rightarrow F(A^*) \le F(A) \text{ in } [0, 0.5) \tag{3.6}$$

and

$$\mu_{A^*}(x_i) \ge \mu_A(x_i) \Rightarrow F(A^*) \le F(A) \text{ in } [0.5, 1). \tag{3.7}$$

From (3.6) and (3.7) we can conclude that $F(A^*) \le F(A)$.

(P-4): It is evident from the definition that $F(A) = F(\overline{A})$ .

Hence (3.3) satisfies all the essential properties of fuzzy information measure. Similarly we can show that $G(A)$ satisfies all the four properties (P-1) to (P-4) of fuzzy information measure.

Since $F(A)$ and $G(A)$ satisfy all the four properties of a valid fuzzy Information measure, therefore we can say that $H_\alpha^\beta(A)$ also satisfy all the four properties of a valid fuzzy Information measure. Hence $H_\alpha^\beta(A)$ is a valid measure of fuzzy Information. We call it a generalized fuzzy Information measure of type $(\alpha, \beta)$.

**Particular cases:**

(i) In case $\alpha = 1$ in (3.1), it reduces to

$$H^\beta(A) = \frac{1}{1-\beta} \sum_{i=1}^{n} \mu_A^\beta(x_i) + \left(1 - \mu_A(x_i)\right)^\beta - 1, \qquad \text{which is (1.8).}$$

(ii) In case $\beta = 1$ in (3.1), it reduces to

$$H^\alpha(A) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \mu_A^\alpha(x_i) + \left(1 - \mu_A(x_i)\right)^\alpha - 1, \text{ which is also (1.8).}$$

(iii) In case $\alpha \to \beta$ in (3.1) and $\alpha, \beta \ne 1$, it reduces to

$$H(A) = -\sum_{i=1}^{n} \mu_A^\alpha(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i))^\alpha \log(1 - \mu_A(x_i))$$

which further reduces to (1.4) when $\alpha = 1$.

## 4. Monotonic Behavior of the Sub Additive Fuzzy Information Measure of Type $\alpha$

In this section we show that (2.1) is a monotonic decreasing function of $\alpha$.
Differentiating (2.1) with respect to $\alpha$, we get

$$\frac{dH_\alpha(A)}{d\alpha} = -2^{\alpha-1}\sum_{i=1}^{n}\left(\mu_A^\alpha(x_i)\left[\log\mu_A(x_i)\right]^2 + (1-\mu_A(x_i))^\alpha\left[\log\left(1-\mu_A(x_i)\right)\right]^2\right)$$
$$-2^{\alpha-1}\sum_{i=1}^{n}\left(\mu_A^\alpha(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))^\alpha\log\left(1-\mu_A(x_i)\right)\right) \quad .(4.1)$$

$$\frac{dH_\alpha(A)}{d\alpha} = -2^{\alpha-1}\sum_{i=1}^{n}\left(\mu_A^\alpha(x_i)\left[\log\mu_A(x_i)\right]\left[1+\log\mu_A(x_i)\right]\right)$$
$$+(1-\mu_A(x_i))^\alpha\left[\log\left(1-\mu_A(x_i)\right)\right]\left[1+\log\left(1-\mu_A(x_i)\right)\right] \quad (4.2)$$

using $1+\log_2\mu_A\left(x_i\right) \geq -\frac{1}{\mu_A\left(x_i\right)}$ in (4.2), we get

$$\frac{dH_\alpha(A)}{d\alpha} = -2^{\alpha-1}\sum -\frac{\mu_A^\alpha(x_i)\left[\log\mu_A(x_i)\right]}{\mu_A(x_i)} - \frac{(1-\mu_A(x_i))^\alpha\left[\log\left(1-\mu_A(x_i)\right)\right]}{(1-\mu_A(x_i))}$$
$$= 2^{\alpha-1}\sum\mu_A^{\alpha-1}(x_i)\left[\log\mu_A(x_i)\right] + (1-\mu_A(x_i))^{\alpha-1}\left[\log\left(1-\mu_A(x_i)\right)\right] \quad (4.3)$$

Now since

$\sum\mu_A^{\alpha-1}(x_i)\left[\log\mu_A(x_i)\right] + (1-\mu_A(x_i))^{\alpha-1}\left[\log\left(1-\mu_A(x_{\leq i})\right)\right] \leq 0$, therefore,

from (4.3) we get $\frac{dH_\alpha(A)}{d\alpha} \leq 0$.

Hence $H_\alpha(A)$ is a monotonic decreasing function of $\alpha$.

**Particular Case:** If $A_F$ is the most fuzzy set i.e. $\mu_{A_F}(x_i) = 0.5 \quad \forall x_i$, then simple calculation shows that $H_\alpha(A_F) = n$ which is independent of $\alpha$ and $\frac{dH_\alpha(A_F)}{d\alpha} = 0$, where for all other fuzzy sets, $H_\alpha(A)$ is a strictly monotonic decreasing function of $\alpha$.

Let A =(0.3,0.4,0.4,0.2) be a standard fuzzy set. We plot the graph of $H_\alpha(A)$ with respect to $\alpha$ where $0.5 \leq \alpha \leq 2$ on the basis of the following computed table:

| $\alpha$ | 0.50 | 0.75 | 1.25 | 1.50 | 2.00 |
|---|---|---|---|---|---|
| $H_{\alpha}(A)$ | 3.952644 | 3.699308 | 3.435157 | 3.363963 | 3.322022 |

**Table 4.1: The values of $H_{\alpha}(A)$ for different values of $\alpha$**

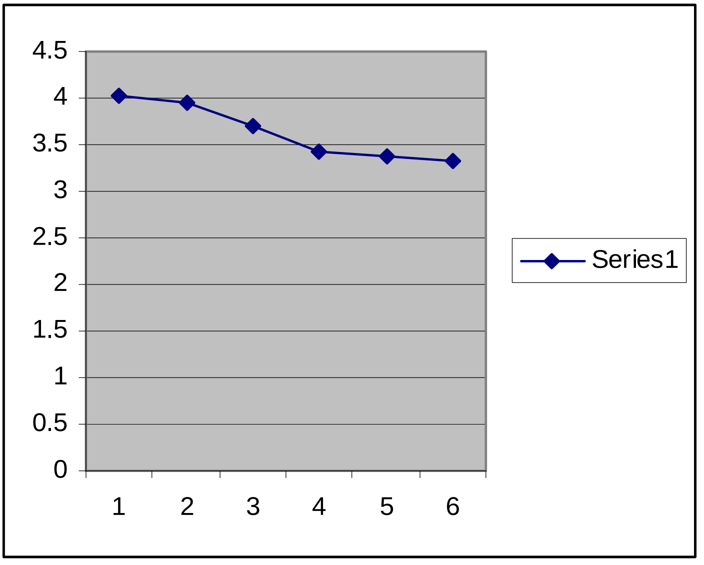


**Fig. 4.1**

It can be concluded from the Table 4.1 and Fig. 4.1 that for $0.5 \leq \alpha \leq 2$, that $H_{\alpha}(A)$ decreases with respect to $\alpha$. Hence in the given range of $\alpha$, $H_{\alpha}(A)$ is a monotonic decreasing function of $\alpha$.

**Comparison Between $H_{\alpha}(P)$ and $H_{\alpha}(A)$ :**

(a) If $P$ is the uniform distribution i.e. $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$ or $P$ is a degenerate function, and then $H_{\alpha}(P)$ given by (1.11) is independent of $\alpha$. Similarly if $A$ is the most fuzzy set or A is crisp set then $H_{\alpha}(A)$ is also independent of $\alpha$.

(b) It can be easily proved that $H_{\alpha}(P)$ is also monotonic decreasing function of $\alpha$. Hence both $H_{\alpha}(P)$ and $H_{\alpha}(A)$ are monotonic decreasing function of $\alpha$.

## 5. Monotonic Behavior of Sub-Additive Fuzzy Information Measure of Type $(\alpha, \beta)$

In this section we study the monotonic behavior of the fuzzy information measure of type $(\alpha, \beta)$ given by (3.1) with respect to $\alpha$ and $\beta$.

Differentiating (3.1) with respect to $\alpha$, we get

$$\frac{dH_\alpha^\beta(A)}{d\alpha}=(\beta-\alpha)^{-2}\mu_A^\beta(x_i)\left[\sum_{i-1}^{n}-\mu_A^{\alpha-\beta}(x_i)\log\left(\mu_A(x_i)\right)^{(\alpha-\beta)}+\mu_A^{\alpha-\beta}(x_i)\right]$$
$$+\left(1-\mu_A(x)\right)^\beta\left[-\left(1-\mu_A(x_i)\right)^{\alpha-\beta}\log\left(1-\mu_A(x_i)\right)^{\alpha-\beta}+\left(1-\mu_A(x_i)\right)^{\alpha-\beta}\right]$$
$$-\mu_A^\beta(x_i)-\left(1-\mu_A(x_i)\right)^\beta \tag{5.1}$$

It implies

$$\frac{(\beta-\alpha)^2\,dH_\alpha^\beta(A)}{d\alpha}=\mu_A(x_i)^\beta\left[\sum_{i-1}^{n}-\mu_A(x_i)^{\alpha-\beta}\log\mu_A(x_i)^{(\alpha-\beta)}+\mu_A(x_i)^{\alpha-\beta}\right]$$
$$+\left(1-\mu_A(x)\right)^\beta\left[-\left(1-\mu_A(x_i)\right)^{\alpha-\beta}\log\left(1-\mu_A(x_i)\right)^{\alpha-\beta}+\left(1-\mu_A(x_i)\right)^{\alpha-\beta}\right]$$
$$-\mu_A(x_i)^\beta-\left(1-\mu_A(x_i)\right)^\beta \tag{5.2}$$

Since $\text{u - ulogu}\leq 1$, therefore

$$(\beta-\alpha)^2\frac{dH_\alpha^\beta(A)}{d\alpha}\leq\sum_{i-1}^{n}\mu_A^\beta(x_i)(1)+\left(1-\mu_A(x_i)\right)^\beta(1)-\mu_A^\beta(x_i)-\left(1-\mu_A(x_i)\right)^\beta\leq 0 \tag{5.3}$$

From (5.3) we can conclude that (3.1) is a decreasing function of $\alpha$. Similarly we can prove expression (3.1) is a decreasing function of $\beta$ also.

Let A = (0.3, 0.4, 0.4, 0.5, 0.2) be a standard fuzzy set. We compute the values of $H_\alpha^\beta(A)$ for different $\alpha$ and $\beta=4$ in the following table:

| $\alpha$ | 0.1 | 0.2 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|
| $H_\alpha^4(A)$ | 1.790805 | 1.638666 | 1.394381 | 1.295971 | 1.210094 | 1.134807 | 1.068498 | 1.009825 |

**Table 5.1: The values of $H_\alpha^\beta(A)$ for different $\alpha$ and $\beta = 4$**

Next we draw the graph of table (5.1) in fig.(5.1).

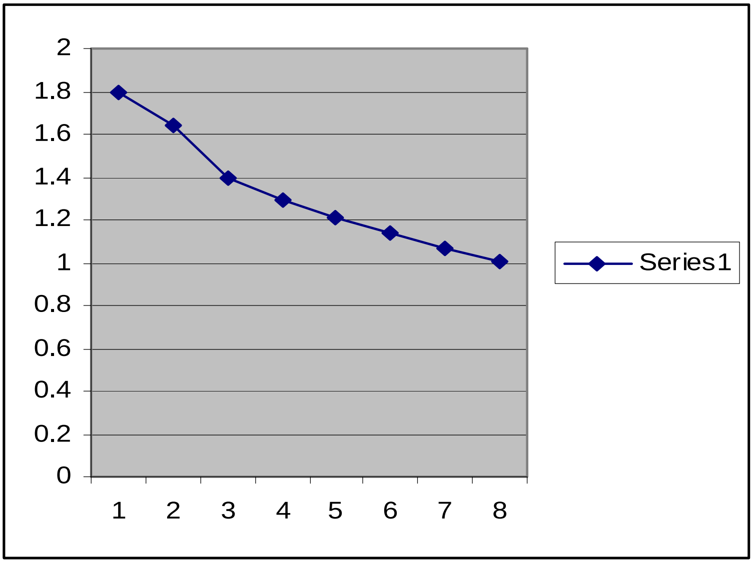


**Fig. 5.1**

From Table (5.1) and Fig.(5.1) we can generalize that $H_\alpha^\beta(A)$ is monotonic decreasing function of $\alpha$ $(0 < \alpha < 1)$ and fixed $\beta > 1$.

Let A = (0.3, 0.4, 0.4, 0.5, 0.2) be a standard fuzzy set. We compute the values of $H_\alpha^\beta(A)$ for different $\alpha$ and $\beta = 0.9$ in the following table:

| $\alpha$ | 10 | 100 | 1000 | 10000 | 100000 |
|---|---|---|---|---|---|
| $H_\alpha^{0.9}(A)$ | 0.48553 | 0.053051 | 0.005262 | 0.000526 | 5.26E-05 |

**Table 5.2: The values of $H_\alpha^\beta(A)$ for different $\alpha$ and $\beta = 0.9$**

Next we draw the graph of table (5.2) in fig.(5.2)

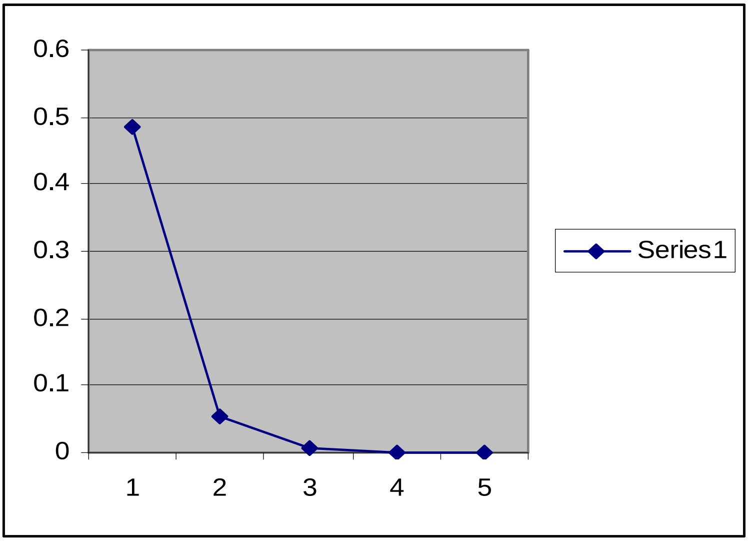


**Fig. 5.2**

From Table 5.2 and Fig. 5.2 we can generalize that the value of $H_{\alpha}^{\beta}(A)$ decreases with respect to $\alpha$ $(\alpha > 1)$ when $\beta < 1$ and $H_{\alpha}^{\beta}(A)$ tends to zero as $\alpha$ tends to infinity.

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