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A NEW LIFETIME DISTRIBUTION FOR MODELING MONOTONIC DECREASING SURVIVAL PATTERNS

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Abstract

One parameter inverse size biased p-dimensional Rayleigh distribution is introduced for modeling lifetimes. Its distributional properties including moments are studied. Hazard function is studied under different parametric settings. Parameter estimation is done using maximum likelihood method and Bayesian approach. Risk of Bayes estimators has been obtained under different loss structures and a comparative risk analysis has been conducted empirically. Based on the findings of simulation study admissible posterior risk functions are identified for four types of loss function.

Key Words: Squared Error, LINEX, Entropy and Precautionary Loss, Maximum Likelihood Estimates, Asymptotically Invariant Prior, Posterior Risk function.

1. Introduction

The Rayleigh distribution (Rayleigh, 1919) is frequently used as a model for the analysis of data resulting from investigation involving wind velocity, wave propagation, radiation and target error by physicists and engineers. Earliest mention of the notion of weighted distributions is found in Fisher (1934). Length or size biased sampling introduced by Cox (1962) is an example of weighted distribution. The p-dimensional Rayleigh distribution was introduced by Cohen and Whitten (1988). In the present article, we propose inverse size-biased p-dimensional Rayleigh distribution (ISBRD). Multiple illustrations on the concept of size biased distributions are found in Patil (2002). These ideas have found wide applicability in disease mapping, survival analysis and in determination of intermediate latency period for a contagion to be detected. Size bias implies that a unit with a large value of the variable has a greater chance of being selected. Size biased distributions come into play when organisms occur in groups, and group size influences the probability of detection (Drummer and McDonald, 1987). For instance, let us consider two variables for investigation of spread of a specific disease, area of a region (say, village) and the number of infected individuals under each area. Proportion of infected people in the i person region represents average from the viewpoint of the infected person which describes size biased average category size. Size biased sampling and modeling is useful for a human population exposed to a contagious microorganism such as Zika or HINI. Such length biased sampling and modeling mechanism can also be applied for example, to the cases of children reported with dyslexia where the disorder exists in different degrees among the affected children and could be corrected through effective and timely intervention.

The rest of the paper is organized as follows. Some statistical characteristics for ISBRD are derived under section two. Parameter estimation under classical and Bayesian approaches and posterior risk analysis under different loss functions, for the complete sample case, is undertaken in section three and four. Empirical investigation of various risk estimators is carried out in section five. Section six summarizes contribution of the present paper.

2. Definition and some distributional properties

Suppose that X is the variable of interest, such that $X \sim f(x; \theta)$. If sample units are weighted with probability proportional to X^{β}, then size-biased density of order β (Patil and Ord, 1976) is represented as $f^{*}_{\beta}(x; \theta) = x^{\beta} f(x; \theta) / \int [x^{\beta} f(x; \theta)]\, dx$.For $\beta = 1$, the distribution is termed as length biased. We propose a size (or length) biased inverse p-dimensional distribution as an alternative lifetime model for survival data. The probability density function of one parameter ISBRD is obtained as

$$f(y;\alpha) = \frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)} \frac{1}{y^{p+2}} \exp\left(-\frac{1}{\alpha y^2}\right);\ y > 0, \alpha > 0, p > 0 \qquad (2.1)$$

We denote ISBRD with parameter α as ISBRD(α). Plot of the density for some values of α are given in Figure 1. A characteristic of the ISBRD(α) is that as parameter α approaches zero, its density function rapidly shifts towards symmetry and becomes less peaked.

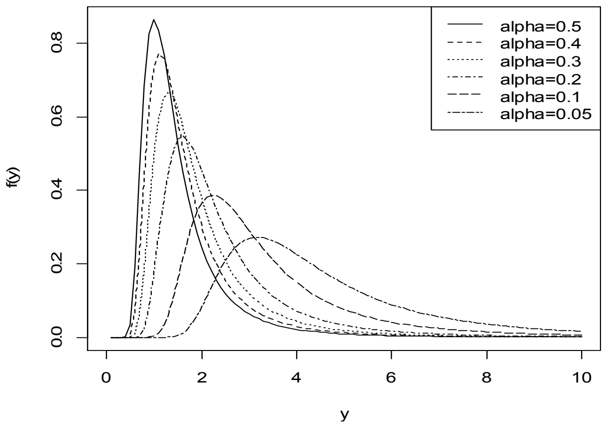


Figure 1: Probability Density function of ISBRD(α)

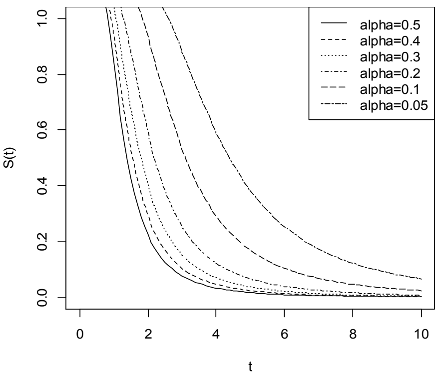
The probability that an individual survives longer than time t, is called survival function. As t ranges from 0 to ∞, the monotone survival curve of the proposed ISBRD (α) goes to 0.The survival function is obtained as a lower incomplete Gamma function.

$$\overline{F}(t) = \frac{1}{\Gamma\left(\frac{p+1}{2}\right)} \gamma\left(\frac{p+1}{2}, \frac{1}{\alpha t^2}\right); t > 0 \qquad (2.2)$$

The hazard function of the proposed ISBRD (α) represents the conditional failure time. It exhibits higher probability of meeting mortality in the beginning of lifetime and takes the following form

$$h(t) = \frac{2}{\alpha^{(p+1)/2}} \frac{1}{t^{p+2}} \exp\left(-\frac{1}{\alpha t^2}\right)\left(\gamma\left(\frac{p+1}{2}, \frac{1}{\alpha t^2}\right)\right)^{-1} \qquad (2.3)$$

Graphs of survival and hazard rate functions under different parametric setting of the parameter α are represented in figures 2 and 3 respectively.



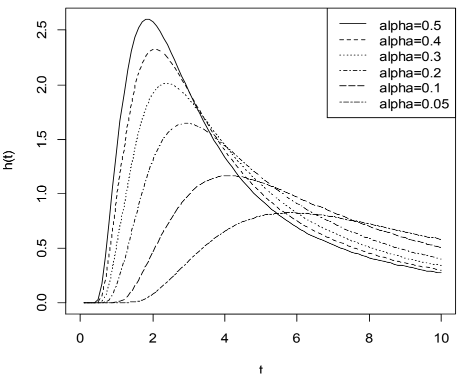


Figure 2: Survival function of ISBRD(α)   Figure 3:Survival function of ISBRD(α)

The survival function is found to be a monotonic decreasing function while the hazard function is seen to increase initially during infancy or early lifetime and then decline at a decreasing pace. This reveals that ISBRD(**α**) is suitable to model lifetimes of the items which have a higher chance of failing during early lifetime, but after survival to a specific development level, the probability of failure decreases as lifetime increases. The proposed distribution, thus, provides flexible alternative model to represent lifetimes with monotonic decreasing survival patterns.

The r^{th}order raw moment is given by $\mu_r' = \frac{\Gamma\left(\frac{p-(r-1)}{2}\right)}{\alpha^{r/2}\Gamma\left(\frac{p+1}{2}\right)}$ and the r^{th} order Central moment is given by $\mu_r = \sum_{i=0}^{r}(-1)^i \binom{r}{i}\left(\frac{\Gamma\left(\frac{p}{2}\right)}{\alpha^{r/2}\Gamma\left(\frac{p+1}{2}\right)}\right)^i \left(\frac{\Gamma\left(\frac{p-(r-i-1)}{2}\right)}{\alpha^{(r-i)/2}\Gamma\left(\frac{p+1}{2}\right)^r}\right)$**.** Mean and variance are respectively obtained as

$$E(y) = \frac{\Gamma\left(\frac{p}{2}\right)}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)}, \quad V(y) = \frac{1}{\alpha\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}\left(\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right) - \left(\Gamma\left(\frac{p}{2}\right)\right)^2\right)$$

Standard Deviation is $\frac{1}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)}\left(\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right) - \left(\Gamma\left(\frac{p}{2}\right)\right)^2\right)^{1/2}$, Coefficient of Variation is given by $\frac{100}{\Gamma\left(\frac{p}{2}\right)}\left(\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)-\left(\Gamma\left(\frac{p}{2}\right)\right)^2\right)^{1/2}$,

Median is $\frac{1}{3}\left(\left(\frac{2}{\alpha(p+2)}\right)^{1/2}+2\frac{\Gamma\left(\frac{p}{2}\right)}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)}\right)$. ISBRD(α) is unimodal with mode at

$\left(\frac{2}{\alpha(p+2)}\right)^{1/2}$ $for\ \alpha>0$.

Table 1 shows empirical values of some important properties corresponding to p=2. The effects of changes on various measures are observed with changing values of parameter α. From this table, it is observed that Mean > Median > Mode. Thus, from Table 1 and Figure 1 it is evident that the proposed ISBRD(*α*) is a positively skewed distribution and variance of ISBRD(α) is a decreasing function of α such that the measure of consistency, coefficient of variation, remains stable at 75.551% for varying values of the parameter α.

α	Mean	Variance	Median	Mode
0.05	5.043	14.535	4.418	3.162
0.1	3.562	7.268	3.124	2.236
0.15	2.916	4.845	2.550	1.826
0.2	2.521	3.634	2.209	1.581
0.25	2.257	2.907	1.976	1.414
0.3	2.061	2.423	1.804	1.291
0.35	1.903	2.076	1.670	1.195
0.4	1.781	1.817	1.562	1.118
0.45	1.682	1.615	1.473	1.054
0.5	1.597	1.454	1.397	1
0.55	1.525	1.321	1.332	0.954
0.6	1.457	1.211	1.275	0.913
0.65	1.395	1.118	1.225	0.877
0.7	1.346	1.038	1.181	0.845
0.75	1.309	0.969	1.141	0.817
0.8	1.265	0.908	1.105	0.791
0.85	1.228	0.855	1.072	0.767
0.9	1.184	0.808	1.041	0.745
0.95	1.156	0.765	1.014	0.726
1	1.124	0.727	0.988	0.707

Table 1: Some statistical features of ISBRD(α) for p=2.

3. Maximum Likelihood Estimation

Let $y_1, y_2,...,y_n$ be a random sample of size *n* from ISBRD(*α*); then the joint likelihood function for complete sample is given by

$$l(y;\alpha)=\frac{2^n}{\alpha^{n(p+1)/2}\left(\Gamma\left(\frac{p+1}{2}\right)\right)^n}\prod_{i=1}^{n}\left(\frac{1}{y_i^{p+2}}\right)\exp\left(-\sum_{i=1}^{n}\left(\frac{1}{\alpha y_i^2}\right)\right) \tag{3.1}$$

The maximum likelihood estimate (MLE) of the scale parameter is $\hat{\alpha}=\frac{2A}{n(p+1)}$ for $A=\sum_{i=1}^{n}\left(\frac{1}{y_i^2}\right)$.

4. Bayesian Estimation

The present paper considers Bayesian estimation of parameter α under four distinct loss functions. The first is squared error loss function (SELF) or quadratic loss, which is classified as a symmetric function which associates equal importance to the losses due to overestimation and underestimation of equal magnitude and is measured as

$$L(\alpha,\hat{\alpha})=(\alpha-\hat{\alpha})^2 \tag{4.1}$$

Symmetric loss functions are not appropriate when overestimation and underestimation are not equally serious. For instance, in construction of dams underestimation of the proposed dam height on a river may result in flooding but overestimation of the dam height does not have dangerous consequence in the event of occurrence of flood or rising of water mark during heavy rains. Asymmetric loss functions deal with the cases where it is more damaging to miss the target on one side than on the other.

The convex loss function known as linear exponential loss function (LINEX), an asymmetric loss function, for a parameter α was introduced by Varian (1975) is

$$L(\Delta)=\left(e^{k\delta}-k\delta-1\right)\ ;\quad k\neq 0 \tag{4.2}$$

where, $\delta=\frac{\hat{\alpha}}{\alpha}-1$ .This asymmetric loss function has been found to be appropriate in the situations where either overestimation is more serious than underestimation and vice-versa. The positive value of *k* is used when overestimation is more serious than underestimation and for negative value of *k,* reverse is true. For *k* close to zero, this loss function is approximately squared error loss and therefore symmetric. These loss functions have been studied by Canfield (1970), Zellner (1986), Rojo (1987), Basu and Ebrahimi (1991) and Soliman (2000) among others.

Another asymmetric loss function, known as General Entropy loss function (GELF) developed by Calabria and Pulcini (1994) is

$$L(\delta)=\left(\delta^d-d\log_e(\delta)-1\right);\ \delta=\frac{\hat{\alpha}}{\alpha}\ and\ d=1 \tag{4.3}$$

Introduced by Norstrom (1996), precautionary loss function (PLF) is also asymmetric in nature, which contains quadratic loss function as a special case. These loss functions approach infinity near the origin to prevent underestimates and thus give conservative estimates, especially when, for example, low failure rates are being estimated. The conservative estimates make these loss functions useful when the consequences are major such as during disease modeling, under estimation of potentiality of an event is disastrous rather than overestimation. It takes the following form

$$L(\hat{\alpha},\alpha)=\frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}} \tag{4.4}$$

Assuming that no information is available, as may happen with a new contagion like Zika, asymptotically invariant prior, proposed by Hartigan (1964) which is of the form $g(\alpha)=\frac{1}{\alpha^3}, \alpha>0$, is adopted for the posterior analysis. In conjunction with the likelihood (3.1) it yields the following posterior density function of ISBRD(α).

$$\Pi(\alpha|\underline{y})=\frac{1}{\alpha^{(n(p+1)+6)/2}}\frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}A^{(n(p+1)+8)/2}e^{-\frac{A}{\alpha}} \tag{4.5}$$

The following posterior analysis is carried out assuming that p is known.

Theorem 1: For a positive integer p and α> 0, under SELF, Bayes estimator of α is the posterior mean

$$\hat{\alpha}_{BS}=\frac{A\Gamma\left(\frac{n(p+1)+6}{2}\right)}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}=\frac{2A}{n(p+1)+6}$$

The posterior risk function of $\hat{\alpha}_{BS}$, under SELF is

$$R_{BS}(\hat{\alpha}_{BS})=\alpha^2\left\{\frac{12\alpha^2(n(p+1)+8)(n(p+1)+10)}{(n(p+1))(n(p+1)+6)^2}+1\right\} \tag{4.6}$$

Theorem 2: For a positive integer p and k> 0, under LINEX loss function, Bayes estimator of αis

$$\hat{\alpha}_{BL}=\frac{A}{k}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]$$

The posterior risk function of $\hat{\alpha}_{BL}$, under SELF is

$$R_{BS}(\hat{\alpha}_{BL})=$$

$$\alpha^2\left[\frac{\alpha^2(n(p+1)+8)(n(p+1)+10)}{k(n(p+1))}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]\left(\left\{\frac{(n(p+1)+12)}{4k}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]\right\}-2\right)+1\right] \tag{4.7}$$

Theorem 3: For a positive integer p and α> 0, under GELF, Bayes estimator of α is

$$\hat{\alpha}_{BE} = \frac{A\Gamma\left(\frac{n(p+1)+8}{2}\right)}{\Gamma\left(\frac{n(p+1)+10}{2}\right)} = \frac{2A}{(n(p+1)+8)}$$

The posterior risk function of $\hat{\alpha}_{BE}$, under SELF is

$$R_{BS}\left(\hat{\alpha}_{BE}\right) = \alpha^2\left\{\frac{8\alpha^2(n(p+1)+10)}{(n(p+1))(n(p+1)+8)}+1\right\} \tag{4.8}$$

Theorem 4: For a positive integer p and α> 0, under PLF, Bayes estimator of α is

$$\hat{\alpha}_{BP} = \frac{A\left[\Gamma\left(\frac{n(p+1)+4}{2}\right)\right]^{1/2}}{\left[\Gamma\left(\frac{n(p+1)+8}{2}\right)\right]^{1/2}} = \frac{2A}{\left[(n(p+1)+4)(n(p+1)+6)\right]^{1/2}}$$

The posterior risk function of $\hat{\alpha}_{BP}$, under SELF is

$$R_{BS}\left(\hat{\alpha}_{BP}\right) = \alpha^2\left[\frac{2\alpha^2(n(p+1)+8)(n(p+1)+10)}{n(p+1)}\left\{\frac{(n(p+1)+12)}{(n(p+1)+4)(n(p+1)+6)} - \frac{1}{\left[(n(p+1)+4)(n(p+1)+6)\right]^{1/2}}\right\}+1\right] \tag{4.9}$$

Posterior risk functions derived in above theorems are found to be functions of the sample data and of the prior parameters.

5. Risk Analysis

We now assess performance of Bayes estimator of the parameter α on the basis of its posterior risk. Comparison in terms of risk helps in identification of a decision rule with the lowest risk. A decision rule is admissible (with respect to the loss function) if and only if no other rule dominates it, otherwise it is inadmissible. A comparison of this type maybe needed to check whether an estimator is inadmissible under some loss function. If it is so, then the estimator would not be used under the specified loss function. For this purpose, risk of the estimators relative to squared error loss have been estimated. It is evident from the expressions of the risks of the estimators that direct comparison among them is not possible. Therefore, an empirical comparison has been made. Using p=2, n=6 for α = 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9, 0.95, 1, in equations (4.6) to (4.9), we compute and report the corresponding risks in Table 2, for complete sample case, so as to observe and compare the behavior of the following risk functions for the proposed lifetime distribution:

α	$R_{BS}(\hat{\alpha}_{BS})$	$R_{BS}(\hat{\alpha}_{BL})$	$R_{BS}(\hat{\alpha}_{BE})$	$R_{BS}(\hat{\alpha}_{BP})$
0.05	0.0025	0.0026	0.0025	0.0025
0.1	0.0101	0.0108	0.0100	0.0103
0.15	0.0229	0.0265	0.0227	0.0245
0.2	0.0413	0.0527	0.0407	0.0462
0.25	0.0657	0.0937	0.0644	0.0775
0.3	0.0968	0.1547	0.0938	0.1212
0.35	0.1351	0.2425	0.1296	0.1802
0.4	0.1815	0.3647	0.1723	0.2584
0.45	0.2371	0.5305	0.2221	0.3602
0.5	0.3026	0.7499	0.2799	0.4903
0.55	0.3796	1.0345	0.3463	0.6543
0.6	0.4692	1.3967	0.4220	0.8583
0.65	0.5729	1.8504	0.5079	1.1089
0.7	0.6923	2.4106	0.6049	1.4133
0.75	0.8291	3.0934	0.7139	1.7792
0.8	0.9851	3.9164	0.8360	2.2151
0.85	1.1623	4.8981	0.9723	2.7298
0.9	1.3628	6.0582	1.1240	3.3329
0.95	1.5887	7.4178	1.2923	4.0346
1	1.8425	8.9992	1.4786	4.8454

Table 2: Posterior risk functions of ISBRD(α) at p=2 and n=6

Table 2 lists the values of risks or average loss over the parameter domain under different loss functions. For $0 < \alpha < 0.15$, all the risks are seen to be equivalent. However, ordering among the four considered risk functions based on the values of $\alpha > 0.15$ are such that risk under GELF < SELF < PLF < LINEX. The risks of the Bayes estimator obtained under GELF, SELF and PLF are much smaller than that obtained under LINEX loss.

5. Conclusion

A new lifetime distribution is proposed to alternatively model lifetime of units which age rapidly or wherein the infection spreads rapidly initially and is arrested under treatment but to different degrees, sometimes resulting in no recovery at all. The theoretical properties of the introduced distribution are derived. Risk analysis is discussed within a Bayes framework. In risk analysis, both the potentiality of an undesired event and its consequences are investigated in terms of the performance of estimators assessed on the basis of their posterior risk which is found to be the least under GELF (Table 2). Therefore, the corresponding risk (GELF) is regarded as the most preferred and is deemed more appropriate, in comparison to other considered loss functions for the proposed life time distribution.

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Appendix

1. Proof of probability density function of ISBRD (α):

The probability density function of the p-dimensional Rayleigh distribution is

$$g(x;\sigma)=\frac{2^{-(p-2)/2}}{\sigma\Gamma\left(\frac{p}{2}\right)}\left(\frac{x}{\sigma}\right)^{p-1}\exp\left(-\frac{1}{2}\left(\frac{x}{\sigma}\right)^{2}\right),\ x>0,\sigma>0 \qquad \text{(i)}$$

Putting $\alpha=2\sigma^2$ in equation (i), we get

$$g(x;\alpha)=\frac{2}{\Gamma\left(\frac{p}{2}\right)}\frac{x^{p-1}}{\alpha^{p/2}}\exp\left(-\frac{x^2}{\alpha}\right),x>0,\alpha>0 \qquad \text{(ii)}$$

where, α is scale parameter.

Mean of p-dimensional Rayleigh distribution is $E(x)=\alpha^{1/2}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}$

Then the probability density function of size biased p-dimensional Rayleigh distribution is obtained as $f(x;\alpha)=\frac{xg(x;\alpha)}{E(x)}$

$$f(x;\alpha)=\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}x^{p}\exp\left(-\frac{x^2}{\alpha}\right),x>0,\alpha>0$$

Thus, Inverse size biased p-dimensional Rayleigh distribution is given by

$$f(y;\alpha)=\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}\frac{1}{y^{p+2}}\exp\left(-\frac{1}{\alpha y^2}\right),y>0,\alpha>0$$

where, $y=1/x$.

2. Proof of Survival function:

$$\overline{F}(t)=\int_{t}^{\infty}f(y;\alpha)dy \qquad ,y>t$$

$$\overline{F}(t)=\int_{t}^{\infty}\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}\frac{1}{y^{p+2}}\exp\left(-\frac{1}{\alpha y^2}\right)dy$$

$$\overline{F}(t)=\frac{1}{\Gamma\left(\frac{p+1}{2}\right)}\int_{0}^{\frac{1}{\alpha t^2}} z^{((p+1)/2)-1}e^{-z}dz \text{ , where } z=\frac{1}{\alpha y^2}$$

$$\overline{F}(t)=\frac{1}{\Gamma\left(\frac{p+1}{2}\right)}\gamma\left(\frac{p+1}{2},\frac{1}{\alpha t^2}\right)$$ whichis the lower incomplete Gamma function.

3. Proof of Hazard rate function:

By definition, $h(t)=\frac{f(t;\alpha)}{\overline{F}(t)}$

or
$$h(t)=\frac{\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}\frac{1}{y^{p+2}}\exp\left(-\frac{1}{\alpha y^2}\right)}{\frac{1}{\Gamma\left(\frac{p+1}{2}\right)}\gamma\left(\frac{p+1}{2},\frac{1}{\alpha t^2}\right)}$$

or $h(t)=\frac{2}{\alpha^{(p+1)/2}}\frac{1}{t^{p+2}}\exp\left(-\frac{1}{\alpha t^2}\right)\left(\gamma\left(\frac{p+1}{2},\frac{1}{\alpha t^2}\right)\right)^{-1}$

4. Proof of expression for the r[th] order raw moment:

$$E(y^r)=\int_0^\infty y^r\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}\frac{1}{y^{p+2}}\exp\left(-\frac{1}{\alpha y^2}\right)dy$$

$$E(y^r)=\frac{1}{\alpha^{r/2}\Gamma\left(\frac{p+1}{2}\right)}\int_0^\infty k^{(p-r-1)/2}e^{-k}dk \text{ , Where, } k=\frac{1}{\alpha y^2}$$

$$\mu_r^{'}=E(y^r)=\frac{\Gamma\left(\frac{p-(r-1)}{2}\right)}{\alpha^{r/2}\Gamma\left(\frac{p+1}{2}\right)}$$

5. Proof of expression for the r[th] order central moment**:**

$$\mu_r = E(y - E(y))^r$$

$$\mu_r = \int_0^\infty (y - E(y))^r f(y, \alpha) dy$$

$$\mu_r = \int_0^\infty \left(y - \frac{\Gamma\left(\frac{p}{2}\right)}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)} \right)^r \frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)} \frac{1}{y^{p+2}} \exp\left(-\frac{1}{\alpha y^2}\right) dy$$

Let $\kappa = \dfrac{\Gamma\left(\frac{p}{2}\right)}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)}$, then

$$\mu_r = \int_0^\infty (y - \kappa)^r \frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)} \frac{1}{y^{p+2}} \exp\left(-\frac{1}{\alpha y^2}\right) dy$$

where, $(y-\kappa)^r = {}^rC_0 y^r (-\kappa)^0 - {}^rC_1 y^{r-1} (-\kappa)^1 + {}^rC_2 y^{r-2} (-\kappa)^2$

or $\mu_r = \int_0^\infty {}^rC_0 y^r (-\kappa)^0 - {}^rC_1 y^{r-1} (-\kappa)^1 + {}^rC_2 y^{r-2} (-\kappa)^2$

or $\mu_r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left(\dfrac{\Gamma\left(\frac{p}{2}\right)}{\alpha^{r/2}\Gamma\left(\frac{p+1}{2}\right)} \right)^i \dfrac{\Gamma\left(\frac{p-(r-i+1)}{2}\right)}{\alpha^{(r-i)/2}\left(\Gamma\left(\frac{p+1}{2}\right)\right)^r}$

6. Proof for Mode:

$$\frac{d}{dy}\left[\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)} \frac{1}{y^{p+2}} \exp\left(-\frac{1}{\alpha y^2}\right) \right] = 0$$

$$y = \left(\frac{2}{\alpha(p+2)} \right)^{1/2}, \alpha > 0$$

7. Proof of posterior density function:
   The likelihood function is given by

$$l(y;\alpha)=\frac{2^n}{\alpha^{n(p+1)/2}\left(\Gamma\left(\frac{p+1}{2}\right)\right)^n}\prod_{i=1}^{n}\left(\frac{1}{y_i^{p+2}}\right)\exp\left(-\sum_{i=1}^{n}\left(\frac{1}{\alpha y_i^2}\right)\right)$$

The posterior density function is

$$\Pi(\alpha|\underline{y})=\frac{l(y;\alpha)g(\alpha)}{\int_0^\infty l(y;\alpha)g(\alpha)d\alpha}\text{, where } g(\alpha)=\frac{1}{\alpha^3},\ \alpha>0$$

$$\Pi(\alpha|\underline{y})=\frac{1}{\alpha^{(n(p+1)+6)/2}}\frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}A^{(n(p+1)+8)/2}e^{-\frac{A}{\alpha}}$$

And

$$f(y;\hat{\alpha})=\frac{1}{\alpha^{(n(p+1)+6)/2}}\frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}\left(\frac{n(p+1)\hat{\alpha}}{2}\right)^{(n(p+1)+2)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)$$

where, the maximum likelihood function (MLE) is $\hat{\alpha}=\frac{2A}{n(p+1)}$

Proof of theorem 1: Under square error loss function (SELF),

$$\hat{\alpha}_{BS}=\int_0^\infty \alpha\frac{1}{\alpha^{(n(p+1)+6)/2}}\frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}A^{(n(p+1)+8)/2}e^{-\frac{A}{\alpha}}d\alpha$$

$$\hat{\alpha}_{BS}=\frac{A\Gamma\left(\frac{n(p+1)+6}{2}\right)}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}=\frac{2A}{n(p+1)+6}$$

The posterior risk function of $\hat{\alpha}_{BS}$, under SELF is

$$R_{BS}(\hat{\alpha}_{BS})=E_\alpha\left(\hat{\alpha}_{BS}^2\right)-2\alpha E_\alpha\left(\hat{\alpha}_{BS}\right)+\alpha^2$$

$$E_\alpha\left(\hat{\alpha}_{BS}^2\right)=\frac{4\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+12)/2}}{\alpha^{(n(p+1)+6)/2}(n(p+1)+6)^2\Gamma\left(\frac{n(p+1)+8}{2}\right)}\int_0^\infty\hat{\alpha}^{(n(p+1)+12)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)d\hat{\alpha}$$

or

$$E_\alpha\left(\hat{\alpha}^2_{BS}\right)=\frac{2\alpha^4(n(p+1)+8)(n(p+1)+10)(n(p+1)+12)}{(n(p+1))(n(p+1)+6)^2}$$

$$E_\alpha\left(\hat{\alpha}_{BS}\right)=\frac{2\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+10)/2}}{\alpha^{(n(p+1)+6)/2}\left(n(p+1)+6\right)\Gamma\left(\frac{n(p+1)+8}{2}\right)}\int_0^\infty \hat{\alpha}^{(n(p+1)+10)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)d\hat{\alpha}$$

or $$E_\alpha\left(\hat{\alpha}_{BS}\right)=\frac{\alpha^3(n(p+1)+10)(n(p+1)+8)}{(n(p+1))(n(p+1)+6)}$$

$$R_{BS}\left(\hat{\alpha}_{BS}\right)=\frac{2\alpha^4(n(p+1)+8)(n(p+1)+10)(n(p+1)+12)}{(n(p+1))(n(p+1)+6)^2}$$
$$-2\alpha\frac{\alpha^3(n(p+1)+8)(n(p+1)+10)}{(n(p+1))(n(p+1)+6)}+\alpha^2$$

$$R_{BS}\left(\hat{\alpha}_{BS}\right)=\alpha^2\left\{\frac{12\alpha^2(n(p+1)+8)(n(p+1)+10)}{(n(p+1))(n(p+1)+6)^2}+1\right\}$$

Proof of theorem 2:

Under Linear Exponential loss function (LINEX),

If $\delta=\frac{\hat{\alpha}}{\alpha}-1$, then the LINEX loss function $L(\delta)$

$$L(\delta)=\left(e^{k\delta}-k\delta-1\right);\ \ k\neq 0$$

$$L(\alpha,\hat{\alpha})=\left\{\exp\left(k\left(\frac{\hat{\alpha}}{\alpha}-1\right)\right)-k\left(\frac{\hat{\alpha}}{\alpha}-1\right)-1\right\}=e^{-k}\left\{\exp\left(k\left(\frac{\hat{\alpha}}{\alpha}\right)\right)-k\left(\frac{\hat{\alpha}}{\alpha}\right)+k-1\right\}$$

The Bayes estimate $\hat{\alpha}$ under the Linex loss function is the solution of

$$\frac{d}{d\alpha}E\{L(\alpha,\hat{\alpha})\}=0$$

$$E\{L(\alpha,\hat{\alpha})\}=e^k\left\{E\left[\exp\left(k\left(\frac{\hat{\alpha}}{\alpha}\right)\right)\right]-kE\left(\frac{\hat{\alpha}}{\alpha}\right)+k-1\right\}$$

Writing $\hat{\alpha}$ as $\hat{\alpha}_{BL}$ we have,

$$\frac{d}{d\hat{\alpha}_{BL}}L(\alpha,\hat{\alpha})=E_p\left\{e^{-k}\frac{\hat{\alpha}_{BL}}{\alpha}\exp\left(k\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right)\right)\right\}-E_p\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right)=0$$

That $\hat{\alpha}_{BL}$ is the solution to the following equation

$$E_p\left\{\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right)e^{k\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right)}\right\}=e^k E_p\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right)$$

The integrating on both sides with respect to α, we get

$$\int_0^\infty\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right)e^{k\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right)}\frac{1}{\alpha^{(n(p+1)+6)/2}}\frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}A^{(n(p+1)+8)/2}e^{-\frac{A}{\alpha}}d\alpha$$

$$=e^k\int_0^\infty\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right)\frac{1}{\alpha^{(n(p+1)+6)/2}}\frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}A^{(n(p+1)+8)/2}e^{-\frac{A}{\alpha}}d\alpha$$

$$\int_0^\infty\frac{1}{\alpha^{(n(p+1)+8)/2}}\exp\left\{-\left(\frac{A-k\hat{\alpha}_{BL}}{\alpha}\right)\right\}d\alpha=e^k\int_0^\infty\frac{1}{\alpha^{(n(p+1)+8)/2}}e^{-\frac{A}{\alpha}}d\alpha$$

$$\hat{\alpha}_{BL}=\frac{A}{k}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]$$

The posterior risk function of $\hat{\alpha}_{BL}$, under SELF is

$$R_{BS}(\hat{\alpha}_{BL})=E_\alpha\left(\hat{\alpha}_{BL}^2\right)-2\alpha E_\alpha(\hat{\alpha}_{BL})+\alpha^2$$

$$E_\alpha\left(\hat{\alpha}_{BL}^2\right)=\frac{\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+12)/2}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]^2}{k^2\alpha^{(n(p+1)+6)/2}\Gamma\left(\frac{n(p+1)+8}{2}\right)}\int_0^\infty\hat{\alpha}^{(n(p+1)+12)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)d\hat{\alpha}$$

$$E_\alpha\left(\hat{\alpha}_{BL}^2\right)=\frac{\alpha^4(n(p+1)+8)(n(p+1)+10)(n(p+1)+12)}{4k^2(n(p+1))}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]^2$$

$$E_\alpha(\hat{\alpha}_{BL})=\frac{\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+10)/2}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]}{k\alpha^{(n(p+1)+6)/2}\Gamma\left(\frac{n(p+1)+8}{2}\right)}\int_0^\infty\hat{\alpha}^{(n(p+1)+10)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)d\hat{\alpha}$$

$$E_\alpha(\hat{\alpha}_{BL})=\frac{\alpha^3(n(p+1)+10)(n(p+1)+8)}{k(n(p+1))}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]$$

$$R_{BS}(\hat{\alpha}_{BL}) = \alpha^2\left[\frac{\alpha^2(n(p+1)+8)(n(p+1)+10)}{k(n(p+1))}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]\left(\left\{\frac{(n(p+1)+12)}{4k}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]\right\}-2\right)+1\right]$$

Proof of theorem 3:

For a positive integer p and α> 0, under GELF, Bayes estimator of α is

$$\hat{\alpha}_{BE} = \left(E_\alpha\left(\frac{1}{\alpha}\right)\right)^{-1}$$

$$\hat{\alpha}_{BE} = \left(\int_0^\infty \frac{1}{\alpha}\frac{1}{\alpha^{(n(p+1)+6)/2}}\frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}A^{(n(p+1)+8)/2}e^{-\frac{A}{\alpha}}d\alpha\right)^{-1}$$

$$\hat{\alpha}_{BE} = \frac{A\Gamma\left(\frac{n(p+1)+8}{2}\right)}{\Gamma\left(\frac{n(p+1)+10}{2}\right)} = \frac{2A}{(n(p+1)+8)}$$

The posterior risk function of $\hat{\alpha}_{BE}$, under SELF is

$$R_{BS}(\hat{\alpha}_{BE}) = E_\alpha(\hat{\alpha}_{BE}^2) - 2\alpha E_\alpha(\hat{\alpha}_{BE}) + \alpha^2$$

$$E_\alpha(\hat{\alpha}_{BE}^2) = \frac{4\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+12)/2}}{\alpha^{(n(p+1)+6)/2}(n(p+1)+8)^2\Gamma\left(\frac{n(p+1)+8}{2}\right)}\int_0^\infty \hat{\alpha}^{(n(p+1)+12)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)d\hat{\alpha}$$

$$E_\alpha(\hat{\alpha}_{BE}^2) = \frac{2\alpha^4(n(p+1)+10)(n(p+1)+12)}{(n(p+1))(n(p+1)+8)}$$

$$E_\alpha(\hat{\alpha}_{BE}) = \frac{2\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+10)/2}}{\alpha^{(n(p+1)+6)/2}(n(p+1)+8)\Gamma\left(\frac{n(p+1)+8}{2}\right)}\int_0^\infty \hat{\alpha}^{(n(p+1)+10)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)d\hat{\alpha}$$

$$E_\alpha(\hat{\alpha}_{BE}) = \frac{\alpha^3(n(p+1)+10)}{(n(p+1))}$$

$$R_{BS}(\hat{\alpha}_{BE}) = \frac{2\alpha^4(n(p+1)+10)(n(p+1)+12)}{(n(p+1))(n(p+1)+8)} - 2\frac{\alpha^4(n(p+1)+10)}{(n(p+1))} + \alpha^2$$

$$R_{BS}(\hat{\alpha}_{BE}) = \alpha^2\left\{\frac{8\alpha^2(n(p+1)+10)}{(n(p+1))(n(p+1)+8)}+1\right\}$$

Proof of theorem 4:

For a positive integer p and α> 0, under Precautionary loss, Bayes estimator of α is

$$\hat{\alpha}_{BE} = \left(E_\alpha\left(\alpha^2\right)\right)^{1/2}$$

$$\hat{\alpha}_{BE} = \left(\int_0^\infty \alpha^2 \frac{1}{\alpha^{(n(p+1)+6)/2}} \frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)} A^{(n(p+1)+8)/2} e^{-\frac{A}{\alpha}} d\alpha\right)^{1/2}$$

$$\hat{\alpha}_{BP} = \frac{A\left[\Gamma\left(\frac{n(p+1)+4}{2}\right)\right]^{1/2}}{\left[\Gamma\left(\frac{n(p+1)+8}{2}\right)\right]^{1/2}} = \frac{2A}{\left[(n(p+1)+4)(n(p+1)+6)\right]^{1/2}}$$

The posterior risk function of $\hat{\alpha}_{BP}$, under SELF is

$$R_{BS}(\hat{\alpha}_{BP}) = E_\alpha\left(\hat{\alpha}_{BP}^2\right) - 2\alpha E_\alpha\left(\hat{\alpha}_{BP}\right) + \alpha^2$$

$$E_\alpha\left(\hat{\alpha}_{BP}^2\right) = \frac{4\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+12)/2}}{\alpha^{(n(p+1)+6)/2}(n(p+1)+4)(n(p+1)+6)\Gamma\left(\frac{n(p+1)+8}{2}\right)} \int_0^\infty \hat{\alpha}^{(n(p+1)+12)/2} \exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right) d\hat{\alpha}$$

$$E_\alpha\left(\hat{\alpha}_{BP}^2\right) = \frac{2\alpha^4(n(p+1)+8)(n(p+1)+10)(n(p+1)+12)}{(n(p+1))(n(p+1)+4)(n(p+1)+6)}$$

$$E_\alpha\left(\hat{\alpha}_{BP}\right) = \frac{2\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+10)/2}}{\alpha^{(n(p+1)+6)/2}\left[(n(p+1)+4)(n(p+1)+6)\right]^{1/2}\Gamma\left(\frac{n(p+1)+8}{2}\right)} \int_0^\infty \hat{\alpha}^{(n(p+1)+10)/2} \exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right) d\hat{\alpha}$$

$$E_\alpha(\hat{\alpha}_{BP}) = \frac{\alpha^3 (n(p+1)+10)(n(p+1)+8)}{(n(p+1))[(n(p+1)+4)(n(p+1)+6)]^{1/2}}$$

$$R_{BS}(\hat{\alpha}_{BP}) = \frac{2\alpha^4 (n(p+1)+8)(n(p+1)+10)(n(p+1)+12)}{(n(p+1))(n(p+1)+4)(n(p+1)+6)} - 2\frac{\alpha^4 (n(p+1)+10)(n(p+1)+8)}{(n(p+1))[(n(p+1)+4)(n(p+1)+6)]^{1/2}} + \alpha^2$$

$$R_{BS}(\hat{\alpha}_{BP}) = \alpha^2\left[\frac{2\alpha^2 (n(p+1)+8)(n(p+1)+10)}{n(p+1)}\left\{\frac{(n(p+1)+12)}{(n(p+1)+4)(n(p+1)+6)} - \frac{1}{[(n(p+1)+4)(n(p+1)+6)]^{1/2}}\right\}+1\right]$$