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THE WEIGHTED XGAMMA DISTRIBUTION: PROPERTIES AND APPLICATION

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Abstract

The weighted xgamma distribution, a weighted version of xgamma distribution (Sen et al. (2016) is introduced and studied in this article. A special non-negative weight function is considered to obtain the form of the weighted xgamma distribution which is shown as a generalization of xgamma distribution. The length biased xgamma distribution is then obtained as a special case of weighted xgamma density. Different distributional and survival properties of length biased xgamma distribution are studied along with the distributions of order statistics and entropy measure. We propose method of moments and maximum likelihood for estimating the unknown parameter of the length biased version. A sample generation algorithm along with a Monte Carlo simulation study is prepared to observe the pattern of the estimates for different sample sizes. Finally, a real life time-to-event data set is analyzed as an illustration and length biased distribution is compared with other standard lifetime distributions and length biased weighted exponential distribution to check the suitability of the model.

Key Words: Weighted Distributions, Maximum Likelihood Estimation, Order Statistics, Fatigue Life.

1. Introduction

The concept of weighted distributions can be traced back to Fisher (1934) in the study of the effect of methods of ascertainment upon the estimation of frequencies. While extending the basic ideas of Fisher, Rao (1965, 1985) saw the need for a unifying concept by identifying various sampling situations that can be modeled by what he termed as weighted distributions. Zelen (1974) introduced weighted distributions to represent what he broadly perceived as length-biased sampling in the context of cell kinetics and the early detection of disease. In a series of articles with other co-authors, Patil has extensively pursued weighted distributions for purposes of encountered data analysis, equilibrium population analysis subject to harvesting and predation, meta-analysis incorporating publication bias and heterogeneity, modeling clustering and extraneous variation, etc. (see for more details on these applications Dennis and Patil (1984), Laird et al. (1988), Patil (1981, 1991, 1996), Patil and Ord (1976), Patil and Rao (1978), Patil and Taillie (1989), Patil et al. (1993), Taillie et al. (1995) and references therein). More references can be seen in Patil (1997).

In this article, we study the weighted version of xgamma distribution as a generalization of xgamma distribution (see Sen et al. (2016)), with special reference study made to its length biased version. The method of moments and maximum likelihood estimation procedures are proposed to estimate the unknown parameter of the length biased xgamma distribution. We apply the length biased xgamma distribution for modeling time-to-event data set.

2. The weighted xgamma distribution

The form of the probability density function (pdf) of weighted distribution, we have by definition, see Patil et al. (1988), as

$$f(x) = \frac{w(x) f_0(x)}{E[w(X)]}. \tag{1}$$

Where w(x) is weight function which is non-negative and $f_0(x)$ is a pdf.
We take here $w(x) = x^r$ for r =1, 2, … , and $f_0(x)$ is the pdf of xgamma distribution (see Sen et al. (2016)), i.e.,

$$f_0(x) = \frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}; x > 0, \theta > 0$$

Note that, when we take $w(x) = x^r$ for r =1, 2, …, then E[w(X)] is nothing but the r^{th} order raw moment of xgamma distribution, hence, $E[X^r] = \frac{r!(\theta + r + a_r)}{\theta^r(1+\theta)}$; where $a_r = a_{r-1} + r$, $r = 1,2,...$ with $a_0 = 0$ and $a_1 = 2$, which simplifies to

$$E[X^r] = \frac{r!}{\theta^r(1+\theta)}\left[\theta + \frac{(1+r)(2+r)}{2}\right]; \text{ for } r = 1,2,....$$

Following (1) we have r^{th} order moment weighted version of xgamma distribution with the following definition.

Definition 1: A non-negative continuous random variable, X, is said to follow weighted xgamma distribution with parameters r and θ if its pdf is of the form

$$f(x) = \frac{2\theta^{r+2}}{r![2\theta + (1+r)(2+r)]}\left(x^r + \frac{\theta}{2}x^{r+2}\right)e^{-\theta x}; x > 0, \theta > 0, r = 1,2,3,.... \tag{2}$$

We denote it by $X \sim WXG(r,\theta)$.
Figure 1 shows the density functions for weighted xgamma distribution for different values of r and θ.

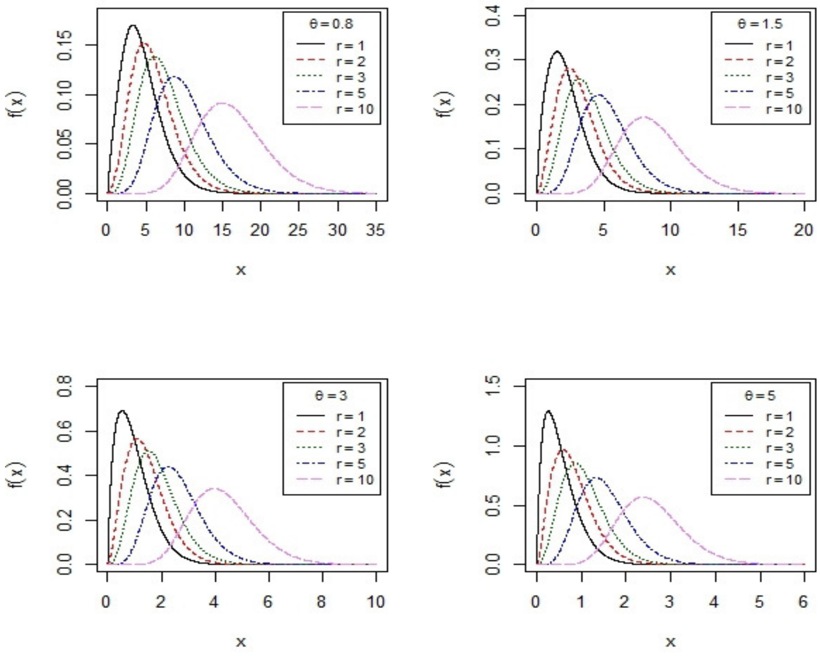

Figure 1: Probability density curves of weighted xgamma distribution for different values of *θ* and *r*.

The k[th] for k =1, 2, 3, …, order raw moment of weighted xgamma distribution in (2), is given by

$$E[X^k] = \frac{(r+k)!}{r!\theta^k}\left[\frac{2\theta+(1+r+k)(2+r+k)}{2\theta+(1+r)(2+r)}\right] \tag{3}$$

The cumulative distribution function (cdf) of weighted xgamma distribution is

$$F(x) = P(X \le x) = \frac{2\theta}{r!\left[2\theta+(1+r)(2+r)\right]}\left[\gamma(r+1,\theta x)+\frac{1}{2\theta}\gamma(r+3,\theta x)\right], \tag{4}$$

where $\gamma(a,x) = \int_0^x u^{a-1}e^{-u}du$ is the lower incomplete gamma function.

The survival function (sf) is then,

$$S(x) = P(X > x) = \frac{2\theta}{r!\left[2\theta+(1+r)(2+r)\right]}\left[\Gamma(r+1,\theta x)+\frac{1}{2\theta}\Gamma(r+3,\theta x)\right], \tag{5}$$

where $\Gamma(a,x)=\int_x^\infty u^{a-1}e^{-u}du$ is the upper incomplete gamma function.

The failure rate (FR) or hazard rate (HR) function is obtained as

$$h(x)=\frac{f(x)}{S(x)}=\frac{\theta^{r+1}\left(x^r+\frac{\theta}{2}x^{r+2}\right)e^{-\theta x}}{\left[\Gamma(r+1,\theta x)+\frac{1}{2\theta}\Gamma(r+3,\theta x)\right]};x,\theta>0,r=1,2,3,... \tag{6}$$

Hereafter, we mainly emphasize on the length biased version of xgamma distribution. The rest of the article is organized as follows:

The length biased version for xgamma distribution is described along with its moments and related measures in section 3. Distributions of order statistics for length biased xgamma distribution are derived in section 4. Entropy measure is described in section 5 and different survival properties are studied in section 6 for length biased version of xgamma distribution. Section 7 deals with the methods of estimation for the unknown parameter in length biased xgamma model. An algorithm for generating random samples from length biased xgamma along with a Monte-Carlo simulation study is presented in section 8. Real data illustration is described in section 9 for studying the application of length biased xgamma model. Finally, the section 10 concludes.

3. Length biased version of xgamma distribution

This section deals with the length biased version of xgamma distribution. The length biased version of xgamma distribution is obtained as a special case of weighted xgamma distribution discussed in the previous section.
If we put r =1 in (2), then we obtain so called length biased version of the xgamma distribution.

Definition 1. A non-negative continuous random variable, X, is said to follow length biased xgamma distribution with parameter θ if its pdf is of the form

$$f(x)=\frac{\theta^3}{(\theta+3)}\left(x+\frac{\theta}{2}x^3\right)e^{-\theta x};x>0,\theta>0. \tag{7}$$

We denote it by $X\sim LBXG(\theta)$.

Note. Length biased xgamma distribution is a special mixture of Gamma (2, θ) and Gamma(4, θ) with mixing proportions θ/(3+θ) and 3/(3+θ), respectively. The probability density plot for different values of θ is shown in figure 2.

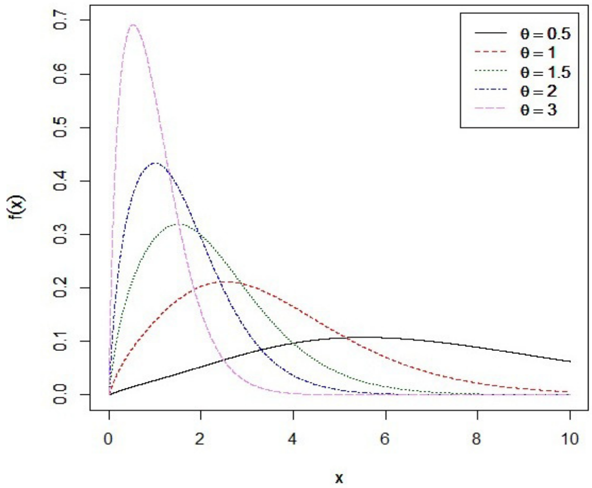

Figure 2: Probability density function of length biased xgamma distribution for different values of θ.

The cdf of $X \sim LBXG(\theta)$ is given by

$$F(x) = 1 - \frac{\left[(3+\theta)+(3+\theta)\theta x+\frac{3}{2}\theta^2x^2+\frac{1}{2}\theta^3x^3\right]}{(\theta+3)}e^{-\theta x}; x>0, \theta>0 \tag{8}$$

The characteristic function (cf) is obtained as

$$\phi_X(x) = E\left[e^{itX}\right] = \frac{\theta^3}{(\theta+3)}\left[(\theta-it)^{-2}+3\theta(\theta-it)^{-4}\right]$$

$i=\sqrt{-1},\ t\in\Re$ (9)

3.1. Moments and associated measures

The k^{th} order raw moments, μ_k' for k=1, 2, 3, …, of length-biased xgamma distribution can be obtained either directly using the pdf in (7) or substituting k =1, 2, 3, …, in (3) after putting r = 1. Hence, we have

$$\mu_k' = E\left(X^k\right) = \frac{(k+1)!\left[2\theta+(2+k)(3+k)\right]}{2\theta^k(\theta+3)} \text{ for } k=1,2,3,... \tag{10}$$

In particular,

$$\mu_1' = E(X) = \frac{2(\theta+6)}{\theta(\theta+3)}; \mu_2' = E\left(X^2\right) = \frac{6(\theta+10)}{\theta^2(\theta+3)}$$

So, we have the expression for second order central (about mean) moment or the population variance for X as

$$V(X)=\mu_2=\frac{2\left(\theta^2+15\theta+18\right)}{\theta^2(\theta+3)^2}$$

so that the coefficient of variation (CV) becomes

$$\gamma=\frac{\sqrt{2\left(\theta^2+15\theta+18\right)}}{2(\theta+6)}$$

The moment generating function (MGF) of *X* is derived as

$$M_X(t)=E\left(e^{tX}\right)=\frac{\theta^3}{(\theta+3)}\left[(\theta-t)^{-2}+3\theta(\theta-t)^{-4}\right]\ ;\ t\in\Re \tag{11}$$

The cumulant generating function (CGF) of *X* is obtained as

$$K_X(t)=\ln M_X(t)=\ln\frac{\theta^3}{(\theta+3)(\theta-t)^2}+\ln\left[1+3\theta(\theta-t)^{-2}\right]\ ;\ t\in\Re \tag{12}$$

4. Distribution of order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from *X ~ LBXG(θ)*. Denote $X_{(j)}$ as the j^{th} order statistic. Then $X_{(1)}$ and $X_{(n)}$ denote the smallest and largest order statistics for a sample of size n drawn from length-biased xgamma distribution with parameter θ, respectively.

The pdf of $X_{(1)}$ is derived as

$$f_{X_{(1)}}(x)=\frac{n\theta^3}{(\theta+3)^n}\left(x+\frac{\theta}{2}x^3\right)\left[(3+\theta)+(3+\theta)\theta x+\frac{3}{2}\theta^2x^2+\frac{1}{2}\theta^3x^3\right]^{n-1}e^{-n\theta x} \tag{13}$$

for $\theta>0, x>0$.

Similarly, the pdf of $X_{(n)}$ is obtained as

$$f_{X_{(n)}}(x)=\frac{n\theta^3}{(\theta+3)^n}\left(x+\frac{\theta}{2}x^3\right)\left[(3+\theta)\left\{1-e^{-\theta x}(1+\theta x)\right\}-\frac{1}{2}\theta^2x^2e^{-\theta x}(3+\theta x)\right]^{n-1}e^{-\theta x} \tag{14}$$

for $\theta>0, x>0$.

5. Renyi entropy measure

The Renyi entropy is defined as

$$H_R(\gamma)=\frac{1}{1-\gamma}\ln\int_0^\infty f^{\gamma}(x)dx \quad \text{for } \gamma>0(\neq 1)$$

When *X ~ LBXG(θ)*, we get

$$\int_0^\infty f^{\gamma}(x)dx=\frac{\theta^{3\gamma}}{(\theta+3)^{\gamma}}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\Gamma(2j+\gamma+1)}{2^j\theta^{j+\gamma+1}\gamma^{2j+\gamma+1}}$$

to obtain Renyi entropy as

$$H_R(\gamma) = \frac{1}{1-\lambda}\left[3\gamma \ln\theta - \gamma \ln(\theta+3)\right] + \frac{1}{1-\lambda}\ln\left[\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\Gamma(2j+\gamma+1)}{2^j \theta^{j+\gamma+1}\gamma^{2j+\gamma+1}}\right] \tag{15}$$

6. Survival properties of length biased xgamma distribution

The survival function for $X \sim LBXG(\theta)$ is

$$S(x) = \frac{\left[(3+\theta)+(3+\theta)\theta x + \frac{3}{2}\theta^2 x^2 + \frac{1}{2}\theta^3 x^3\right]}{(\theta+3)} e^{-\theta x}; x,\theta > 0 \tag{16}$$

Hence, the hazard rate/failure rate function is obtained as

$$h(x) = \frac{\theta^3\left(x + \frac{\theta}{2}x^3\right)}{\left[(3+\theta)+(3+\theta)\theta x + \frac{3}{2}\theta^2 x^2 + \frac{1}{2}\theta^3 x^3\right]}; x,\theta > 0 \tag{17}$$

The hazard rate plot for different values of θ is shown in the figure 3. It is clear that the hazard rate is increasing function in x (> 0). The fact can easily be identified as the length biased distribution given in (7) is log-concave. Therefore, the distribution posses increasing failure rate (IFR) and decreasing mean residual life (DMRL) property.

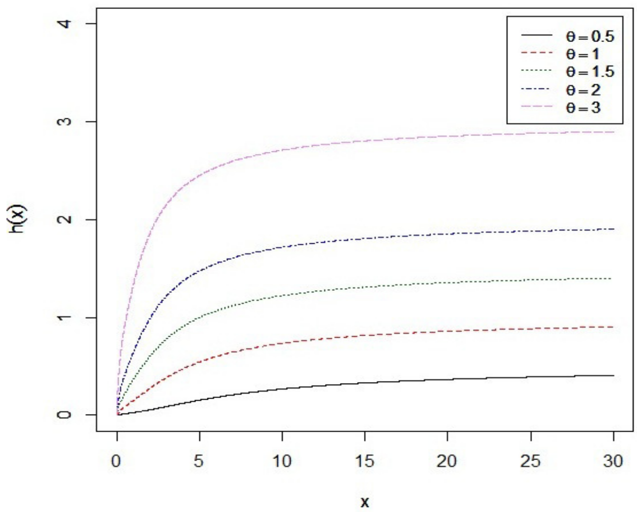

Figure 3: Hazard rate function of length biased xgamma distribution for different values of θ.

For a continuous random variable X with pdf f(x) and cdf F(x), the mean residual life (MRL) function is defined as

$$m(x) = E(X - x \mid X > x) = \frac{1}{1 - F(x)} \int_x^{\infty} [1 - F(t)]dt$$

When $X \sim LBXG(\theta)$, the MRL function is obtained as

$$m(x) = \frac{1}{\theta} + \frac{(\theta+3) + 6(1+\theta x) + \frac{3}{2}\theta^2 x^2}{\theta\left[(3+\theta) + (3+\theta)\theta x + \frac{3}{2}\theta^2 x^2 + \frac{1}{2}\theta^3 x^3\right]} \tag{18}$$

It is to be noted that the MRL function in (18) is bounded below by 1/θ and bounded above by 2(θ+6)/θ(θ+3)=E(X) and is decreasing in x. The plot of MRL function for different values of θ is shown in figure 4.

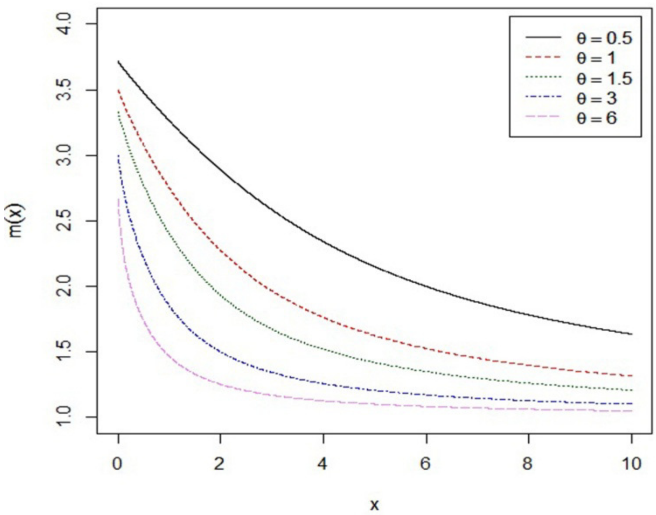

Figure 4: Mean residual life function of length biased xgamma distribution for different values of θ.

Theorem 1. If X ~ LBXG(θ_1) and Y ~ LBXG(θ_2), then for $\theta_1 > \theta_2$, X is smaller than Y in hazard rate order (i.e., $X \leq_{hr} Y$) and thereby in mean residual life order (i.e., $X \leq_{mrl} Y$) and stochastic order (i.e., $X \leq_{st} Y$), respectively.

Proof. For t > 0, we have the ratio of the hazard functions of X and Y as

$$\frac{h_X(t)}{h_Y(t)} = \left(\frac{\theta_1}{\theta_2}\right)^3 \left(\frac{2+\theta_1 t^2}{2+\theta_2 t^2}\right) \left[\frac{(3+\theta_2) + (3+\theta_2)\theta_2 t + \frac{3}{2}\theta_2^{\,2} t^2 + \frac{1}{2}\theta_2^{\,3} t^3}{(3+\theta_1) + (3+\theta_1)\theta_1 t + \frac{3}{2}\theta_1^{\,2} t^2 + \frac{1}{2}\theta_1^{\,3} t^3}\right]$$

which is more than unity if $\theta_1 > \theta_2$ (see figure5 for the plots of $h_X(t)/h_Y(t)$ for selected values of $\theta_1$and θ_2). Hence, $h_X(t) > h_Y(t)$ for $\theta_1 > \theta_2$ and $t > 0$. So, $X \leq_{hr} Y$.

Again, we know that $X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y$ and $X \leq_{hr} Y \Rightarrow X \leq_{st} Y$, and hence the proof.

The reversed hazard rate function of *X ~ LBXG(θ)* is given by (see figure 6 for the plots of reversed hazard rate function for selected values of *θ*)

$$r(x) = \frac{\theta^3\left(x + \frac{\theta}{2}x^3\right)e^{-\theta x}}{(\theta+3)\left\{1-(1+\theta x)e^{-\theta x}\right\} - \frac{1}{2}\theta^2 x^2(3+\theta x)e^{-\theta x}}\ ;\ x > 0, \theta > 0 \tag{19}$$

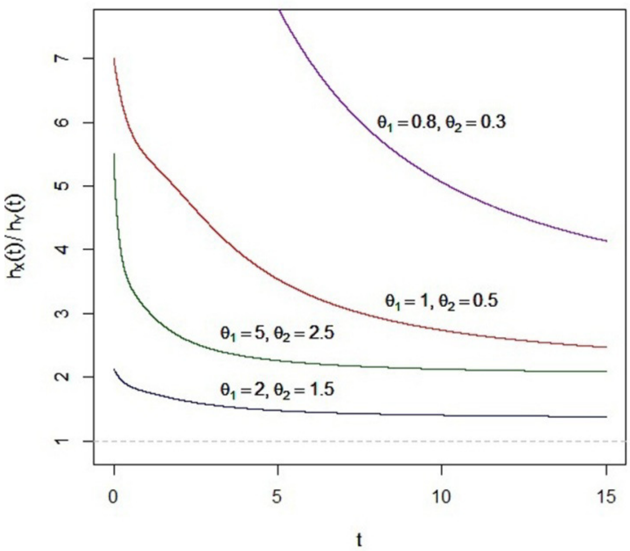

Figure 5: Plots for $h_X(t)/h_Y(t)$ for selected values of θ_1 and θ_2 ($\theta_1 > \theta_2$).

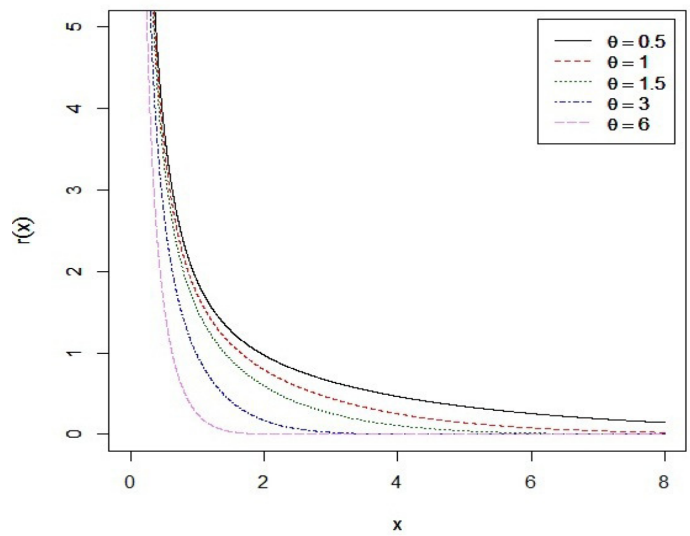

Figure 6: Reversed hazard rate function of length biased xgamma distribution for different values of θ.

7. Estimation of the parameter

In this section we propose method of moments and maximum likelihood estimators for θ when X ~ *LBXG(θ)*. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from *LBXG(θ)*.

7.1 Method of moment estimator

If $\overline{X}$ denotes the sample mean, then by applying the method of moments, we have

$$\overline{X} = \frac{2(\theta+6)}{\theta(\theta+3)}$$

Now, if we denote $\hat{\theta}_M$ as the method of moment estimator for θ, then we have

$$\hat{\theta}_M = \frac{-(3\overline{X}-2)+\sqrt{(3\overline{X}-2)^2+48\overline{X}}}{2\overline{X}} \quad \text{for } \overline{X} > 0 \tag{20}$$

7.2 Maximum likelihood estimator

Let $\tilde{x} = (x_1, x_2, \ldots, x_n)$ be sample observation on $X_1, X_2, \ldots, X_n$. The likelihood function of θ given $\tilde{x}$ is then written as

$$L(\theta \mid \tilde{x}) = \prod_{i=1}^{n} \frac{\theta^3}{(\theta+3)}\left(x_i + \frac{\theta}{2} x_i^{\,3}\right)\exp(-\theta x_i) = \frac{\theta^{3n}}{(\theta+3)^n}\exp\left(-\theta\sum_{i=1}^{n} x_i\right)\prod_{i=1}^{n}\left(x_i + \frac{\theta}{2} x_i^{\,3}\right)$$

The log-likelihood function is given by

$$L(\theta \mid \tilde{x}) = 3n\ln\theta - n\ln(\theta+3) - \theta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\ln\left(x_i + \frac{\theta}{2}x_i^{\,3}\right) \tag{21}$$

Differentiating (21) with respect to θ and equating with zero, we have the log-likelihood equation as

$$\frac{3n}{\theta} - \frac{n}{(\theta+3)} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\left(\frac{x_i^{\,2}/2}{1+\frac{\theta}{2}x_i^{\,2}}\right) = 0 \tag{22}$$

Differentiating (21) twice with respect to θ, we have

$$\frac{\partial^2}{\partial\theta^2}L(\theta \mid \tilde{x}) = \frac{n}{(\theta+3)^2} - \frac{3n}{\theta^2} - \sum_{i=1}^{n}\left(\frac{x^2{}_i/2}{1+\frac{\theta}{2}x^2{}_i}\right)^2 \tag{23}$$

The equation (22) cannot be solved analytically; hence for finding the maximum likelihood estimator for θ we apply numerical method.

8. Sample generation and simulation study

Now we discuss the procedure for simulating random sample of specific size from length-biased xgamma distribution given in (7). We make use of the fact that length-biased xgamma distribution is a special mixture of Gamma (2,θ) and Gamma (4,θ) with mixing proportions θ/(3+θ) and 3/(3+θ), respectively, for constructing the simulation algorithm from the distribution. If *X ~ LBXG(θ)*, then for generating a random sample of size n, $X_1, X_2, \ldots, X_n$, we have the following algorithm:

1. Generate $U_i \sim uniform\ (0,1); i = 1,2,\ldots,n.$
2. Generate $V_i \sim gamma\ (2,\theta); i = 1,2,\ldots,n.$
3. Generate $W_i \sim gamma(4,\theta); i = 1,2,\ldots,n.$
4. If $U_i \le \frac{\theta}{\theta+3}$, then set $X_i = V_i$, otherwise set $X_i = W_i$.

A Monte-Carlo simulation study was carried out considering N=10,000 times for selected values of n and θ. Samples of sizes 20, 40, 60 and 100 were considered and values of θ were taken as 0.1, 0.5, 1.0, 1.5, 3, 4.5 and 6.

The required numerical evaluations are carried out using R software. The following two measures were computed:

(i) Average estimate of θ:

$$\hat{\theta} = \frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_i \text{, where } \hat{\theta}_i\text{'s are simulated estimates.}$$

(ii) Mean Square Error (MSE) of the simulated estimates $\hat{\theta}_i$, $i=1, 2, \ldots, N$:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2 .$$

The result of the simulation study is presented in Table 1. The following observations are made from the simulation study:

(i) For a given value of θ, the average mean square error (MSE) decreases as sample size n increases.

(ii) For a larger given value of θ, MSE gets higher and follow the similar trends as indicated in (i) above.

9. Application with real data illustration

In this section we analyze a real life data set to illustrate the applicability of length biased xgamma distribution.

Fatigue is an important factor in determining the service life of ball bearings. Bearing manufacturers are therefore constantly engaged in fatigue-testing operations in order to obtain information relating fatigue life to load and other factors.

	n = 20		n = 40	
θ	Estimate	MSE	Estimate	MSE
0.1	0.09989	0.00013	0.09976	0.00006
0.5	0.48862	0.00326	0.48647	0.00171
1.0	0.95012	0.01445	0.94499	0.00884
1.5	1.39521	0.03706	1.38425	0.02610
3.0	2.64595	0.22346	2.62256	0.18895
4.5	3.80880	0.68941	3.78011	0.61890
6.0	4.91580	1.53158	4.88066	1.42160
	n = 60		n = 100	
θ	Estimate	MSE	Estimate	MSE
0.1	0.09963	0.00004	0.09935	0.00002
0.5	0.48404	0.00126	0.48376	0.00085
1.0	0.94193	0.00714	0.94072	0.00576
1.5	1.38165	0.02240	1.38010	0.01935
3.0	2.61816	0.17740	2.61517	0.16657
4.5	3.76977	0.59767	3.76117	0.58579
6.0	4.85885	1.41716	4.85344	1.38345

Table 1: Estimate and average MSE for different sample sizes

We use a data set of 23 fatigue life for deep-groove ball bearings, compiled by American Standards Association and reported in Lieblein and Zelen (1956) to illustrate the applicability of our proposed length biased xgamma model. The data set (given in Table 2 is positively skewed (skewness = 0.94 and kurtosis = 0.49) with mean value 72.22, median 67.80 and is unimodal (mode at 50).

17.88,	28.92,	33.00,	41.52,	42.12,	45.60,	48.48,	51.84,	51.96,	54.12,	55.56,	67.80,
68.64,	68.64,	68.88,	84.12,	93.12,	98.64,	105.12,	105.84,	127.92,	128.04,	173.40.	

Table 2: Fatigue lives of 23 deep-groove ball bearings

For comparison purpose, besides length biased xgamma distribution with parameter θ, five other different lifetime distributions, viz., exponential with rate θ, gamma distribution with shape α and rate θ, weibull distribution with shape α and scale β, xgamma distribution with parameter θ and length biased weighted exponential distribution with parameters α and λ, i.e., *LBWE(α, λ)* (see Das and Kundu (2016)), are considered.

In order to compare we consider criteria like, -log-likelihood, Akaike information criterion (AIC) and Bayesian information criterion (BIC), for the data set.

$$\text{AIC} = -2\ln(\textit{likelihood}) + 2k;$$
$$\text{BIC} = k\ln(n) - 2\ln(\textit{likelihood}),$$

where ln(*likelihood*) denotes the log-likelihood function evaluated at the maximum likelihood estimate, *k* is the number of parameters and *n* is the sample size.

The better fitted distribution corresponds to smaller -log-likelihood, AIC and BIC values. We use maximum likelihood method of estimation (MLE) for estimating the model parameter(s). Statistical software R is utilized for computation. Table 3 shows the maximum likelihood estimates (MLEs) of the model parameter(s) with standard error(s) of estimates in parenthesis (Std. Error) and model selection criteria. The figure 7 shows the plot of histogram and fitted exponential, gamma, weibull, xgamma and length biased xgamma curves for fatigue lives data.

10. Concluding remarks

The weighted xgamma distribution is proposed and studied in this article as a generalization of xgamma distribution, which serves as a useful lifetime model in describing time-to-event data sets. As a special case of weighted xgamma distribution, length biased version of xgamma distribution is obtained and its properties are studied in detail. We observe that length biased xgamma distribution is a potential model in describing real life time-to-event data and can be utilized as a flexible model against the standard lifetime models available in the literature. We expect that the proposed weighted xgamma distribution along with the length biased version of xgamma distribution will serve as a competitive model, as reflected by its delegate distributional and survival properties, in describing data coming from survival and reliability fields as well as other fields of application.

Distributions	Estimate (Std. Error)	-Log-likelihood	AIC	BIC
Exponential(θ)	$\hat{\theta}$= 0.0138 (0.0029)	121.435	244.870	246.005
Gamma(α, θ)	$\hat{\alpha}$ = 4.0260 (1.1396) $\hat{\theta}$= 0.0557 (0.0168)	113.029	230.059	232.330
Weibull(α, β)	$\hat{\alpha}$ = 2.1021 (0.3286) $\hat{\beta}$ = 81.8683 (8.5986)	113.691	231.383	233.654
Xgamma(θ)	$\hat{\theta}$= 0.0407 (0.0049)	113.966	229.931	231.067
LBWE(α, λ)	$\hat{\alpha}$ = 0.0251 (0.8960) $\hat{\lambda}$ = 0.0410 (0.0182)	113.522	231.045	233.326
LBXG(θ)	$\hat{\theta}$= 0.0549 (0.0057)	113.086	228.171	229.307

Table 3: MLEs of model parameters and model selection criteria for fatigue lives of ball bearing data

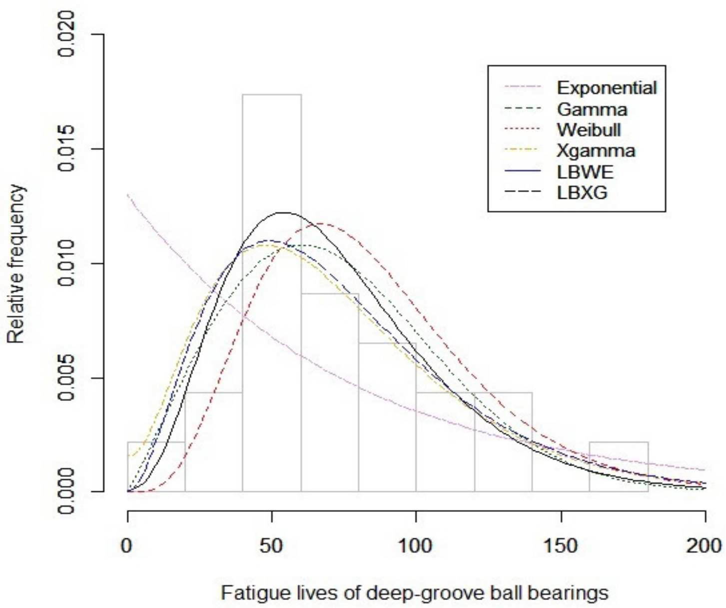

Figure 7: Plot of histogram and fitted exponential, gamma, weibull, xgamma, length biased weighted exponential and length biased xgamma curves for fatigue lives data.

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