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RELIABILITY CALCULATION OF A PARALLEL REDUNDANT SYSTEM WITH DIFFERENT FAILURE RATE&REPAIR RATE USING MARKOV MODELING

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Abstract

The intent of this paper is to discuss a reliability calculation technique, using Markov modeling of a parallel redundant system which can be repairable or non-repairable. In this paper, we will use Markov Modeling technique to provide the derivation for the mean-time-to-failure (non-repairable system) or mean-time-between-failure (repairable system) of a parallel redundant system, with different unit failure or repair rate, to evaluate the reliability and dependability of a parallel operative redundant system.

Key Words: Markov Modeling, Parallel Redundancy, Repairable, Non-repairable, State Transition Diagram, Transition Matrix, MTTF, MTBF.

1. Introduction

In this paper, we introduce a reliability calculation methodology using Markov Modeling, and apply it to the reliability calculation of a parallel redundant system with different failure rate and repair rate. In the reliability regime, systems generally can be grouped into two categories: repairable or non-repairable. For repairable systems the measure of interest is often mean-time-between-failure (MTBF) and for non-repairable systems (e.g. Space system), the measure of interest is often mean-time-to-failure (MTTF) (the first failure).

Markov Modeling is a useful and powerful analysis technique with applications in systems which vary discretely or continuously with respect to time (or space in some case). In reliability we are generally concerned with continuous time, discrete state models. These systems are characterized by randomly varying stochastic processes. Stochastic processes must have two important properties in order for them to be modelled with the Markov approach: 1) memoryless; 2) stationary. A memoryless system is characterized by the fact that the future state of the system depends only on its present state; a stationary system is one in which the probabilities which govern the transitions from state to state remain constant with time (i.e. constant failure rate or repair rate).

Markov Modeling utilizes state transition diagrams to describe the various sequences of states, and rates of transitions between states to calculate or approximate the probability of being in any particular system state. The states of the model are defined by the system unit failures and how they are maintained. The transitional probabilities between states are a function of the failure rates and repair rates (for repairable system) of the various system units. A set of first-order differential equations are developed by describing the probability of being in each state in terms of the transitional probabilities from and to each state. The number of first-order differential equations will equal the number of states of the model. The Markov model is implemented to determine the probability of being in a given discrete state at a given time. By adding up the probabilities of discrete states that represents system success, the reliability of the system can be determined.

2. Application of Coverage to Markov Modeling

In redundant system modeling we generally consider three Markov element states: good, failed covered, and failed uncovered. Covered and uncovered markov element states are mutually exclusive meaning that an element cannot fail both covered and uncovered. System coverage is generally defined in terms of the conditional probability.

When computing coverage for Markov model elements we are concerned with that portion of the Markov element failure rate that is detectable and isolatable. Reconfiguration becomes a function of what resources are available at the time the failure occurs.

Figure 1 gives an example of how coverage is used in the Markov model. In this case if either fan fails covered, the other fan has the ability to take over full cooling function. However, if either fan fails uncovered, system failure occurs. The Markov model for this example appears in Figure 2. Note that once state two is entered, no resources are available and both the covered and uncovered portions of the remaining fans failure rate are routed to system failure.

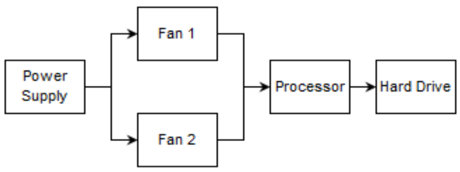

Figure 1: **Markov Model Elements Example**

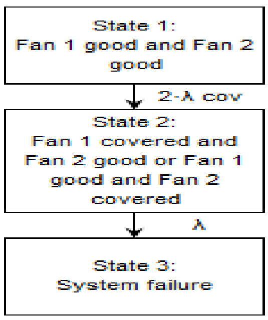

Figure 2: **Coverage Example**

3. Non-repairable parallel redundant units with different failure rate

In order to illustrate how the Markov model equations are developed, we assume an operative redundant system which is made of two units as shown in Figure 3. System success is defined as one of the two units (unit 1 or unit 2) must be working. In the following model, state one is the initial state where unit 1 and unit 2 are both operating properly. State two and three are the states where one unit has failed, the remaining unit is still working to keep the system operational (success). System only fails if both unit 1 and unit 2 fail to meet the system operational requirement. State four is reached when unit 1 and unit 2 have both failed. An assumption used in developing state transition diagram is that unit 1 and unit 2 cannot change states simultaneously.

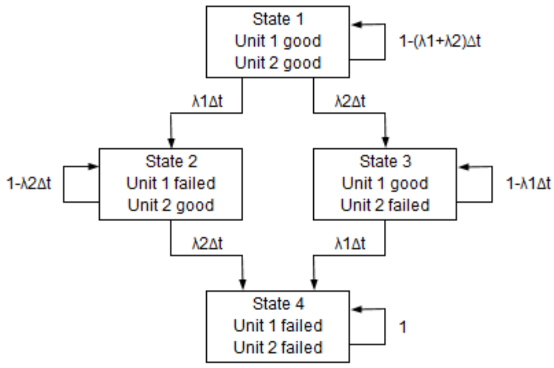

Figure 3: **Non-repairable system state transition diagram**

For state one:

The probability of being in state one at time t+Δt is equal to the probability of being in state one at time t and not transitioning out during Δt. This can be written as:

$$P_1(t+\Delta t) = P_1(t)\cdot\left[1-(\lambda_1+\lambda_2)\Delta t\right] \tag{1}$$

Rearranging by moving P_1 (t) from the right-hand side to left-hand side, and dividing Δt on the both sides of equation (1) to obtain equation (2):

$$\frac{P_1(t+\Delta t)-P_1(t)}{\Delta t} = \frac{dP_1(t)}{dt} = -(\lambda_1+\lambda_2)\cdot P_1(t) \tag{2}$$

Taking Laplace transform to transform the differential equation (2) into the algebraic equation (3):

$$sP_1(s)-P_1(0) = -(\lambda_1+\lambda_2)P_1(s) \tag{3}$$

The boundary condition at time equal zero P_1 (0) =1 (both units are operational):

$$sP_1(s)-1 = -(\lambda_1+\lambda_2)P_1(s) \tag{4}$$

$$P_1(s)=\frac{1}{s+(\lambda_1+\lambda_2)} \tag{5}$$

Taking the inverse Laplace transform for P_1 (s) in equation (5) to obtain P_1 (t) in equation (6):

$$P_1(t)=e^{-(\lambda_1+\lambda_2)t} \tag{6}$$

For state two:

The probability of being in state two at time t+Δt is equal to the probability of being in state one at time t and transitioning to state two in Δt plus the probability of being in state two at time t and not transitioning out during Δt. This can be written as:

$$P_2(t+\Delta t)=P_1(t)\cdot\lambda_1\cdot\Delta t+P_2(t)\cdot(1-\lambda_2\Delta t) \tag{7}$$

Rearranging by moving P_2 (t) from the right-hand side to left-hand side, and dividing Δt on the both sides of equation (7) to obtain equation (8):

$$\frac{P_2(t+\Delta t)-P_2(t)}{\Delta t}=\frac{dP_2(t)}{dt}=\lambda_1\cdot P_1(t)-\lambda_2\cdot P_2(t) \tag{8}$$

Taking Laplace transform to transform the differential equation (8) into the algebraic equation (9):

$$sP_2(s)-P_2(0)=\lambda_1\cdot P_1(s)-\lambda_2\cdot P_2(s) \tag{9}$$

The boundary condition of P_2 (0) =0, and substitute into the equation (9):

$$sP_2(s)=\lambda_1 P_1(s)-\lambda_2 P_2(s) \tag{10}$$

$$P_2(s)=\frac{\lambda_1 P_1(s)}{s+\lambda_2}=\frac{\lambda_1}{(s+\lambda_2)(s+\lambda_1+\lambda_2)}=\frac{1}{s+\lambda_2}-\frac{1}{s+\lambda_1+\lambda_2} \tag{11}$$

Taking the inverse Laplace transform for P_2 (s) to obtain P_2 (t):

$$P_2\,(t)=e^{-\lambda_2 t}-e^{-(\lambda_1+\lambda_2)t} \tag{12}$$

The same approach applies to the state three probability equation P_3 (t):

$$P_3(t)=e^{-\lambda_1 t}-e^{-(\lambda_1+\lambda_2)t} \tag{13}$$

The reliability function depends on the state one, state two and state three as we define the system success is at least one of two units must be working:

$$R(t)=P_1(t)+P_2(t)+P_3(t)=e^{-(\lambda_1+\lambda_2)t}+e^{-\lambda_2 t}+e^{-\lambda_1 t}-2e^{-(\lambda_1+\lambda_2)t} \tag{14}$$

Simplifying to obtain the equation (15):

$$R(t) = e^{-\lambda_2 t} + e^{-\lambda_1 t} - e^{-(\lambda_1 + \lambda_2)t} \qquad (15)$$

The Mean Time To Failure (MTTF) is the integral from time zero to infinity of the reliability function of the system. Here, the reliability of the system is defined by state one, state two and state three. State four is the failed condition. Consequently, we can write the MTTF:

$$MTTF = \int_0^\infty R(t)dt = \int_0^\infty (e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t})dt = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2} \qquad (16)$$

This MTTF equation has the same result as the equation 5 in Table 6.2 -3, Page 161, Reliability Toolkit: Commercial Practices Edition published by Reliability Analysis Center [4]:

$$\lambda_{1/2} = \frac{\lambda_A{}^2 \cdot \lambda_B + \lambda_B{}^2 \cdot \lambda_A}{\lambda_A{}^2 + \lambda_B{}^2 + \lambda_A \cdot \lambda_B}$$

The difference between these two equations are: the equation (16) calculates the expected time (MTTF) whereas equation 5 calculates the effective failure rate λ.

The above-mentioned MTTF equation (16) can also be verified by the following Binomial distribution reliability modeling process. The basic steps involved in developing this type of reliability modeling are to first define what is required for mission success and second to define the probability of being in each possible operating state (good or failed). The probability of successful operation is the sum of the probability of being in a good state.

State	Unit 1	Unit 2	System state
1	Good	Good	Success
2	Failed	Good	Success
3	Good	Failed	Success
4	Failed	Failed	down

Table 1: **System operating state**

The reliability function in the exponential case is $R(t) = e^{-\lambda t}$, where λ is the failure rate and t is the period of time over which reliability is measured. The probability of failure (Unreliability) is equal to $1 - R(t) = 1 - e^{-\lambda t}$. Then $e^{-\lambda t}$ and $1 - e^{-\lambda t}$ are substituted in the above Table 1 and simplified as shown in the following Table 2:

State	Unit 1	Unit 2	Probability of being in state
1	$e^{-\lambda_1 t}$	$e^{-\lambda_2 t}$	$e^{-\lambda_1 t}\cdot e^{-\lambda_2 t}$
2	$e^{-\lambda_1 t}$	$1-e^{-\lambda_2 t}$	$e^{-\lambda_1 t}\,.\,(1-e^{-\lambda_2 t})$
3	$1-e^{-\lambda_1 t}$	$e^{-\lambda_2 t}$	$(1-e^{-\lambda_1 t})\cdot e^{-\lambda_2 t}$
4	$1-e^{-\lambda_1 t}$	$1-e^{-\lambda_2 t}$	$(1-e^{-\lambda_1 t})\cdot(1-e^{-\lambda_2 t})$

Table 2: Probability of being in state

The probability of successful system operation is equal to the sum of probability of being in state1, 2 and 3 in the above Table 2, and is expressed as:

$$P(Success)=P(1)+P(2)+P(3)$$

$$\begin{aligned}P(Success)&=e^{-\lambda_1 t}\cdot e^{-\lambda_2 t}+e^{-\lambda_1 t}\cdot(1-e^{-\lambda_2 t})+(1-e^{-\lambda_1 t})\cdot e^{-\lambda_2 t}\\&=e^{-\lambda_1 t}+e^{-\lambda_2 t}-e^{-(\lambda_1+\lambda_2)t}\end{aligned}\tag{17}$$

We can write the MTTF:

$$MTTF=\int_0^\infty P(Success)dt=\int_0^\infty (e^{-\lambda_1 t}+e^{-\lambda_2 t}-e^{-(\lambda_1+\lambda_2)t})dt=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}-\frac{1}{\lambda_1+\lambda_2}\tag{18}$$

We notice that equation (18) is the same as equation (16).

Note that when unit 1 and unit 2 have the same failure rate, we have the following familiar equation:

$$MTTF=\frac{3}{2\lambda}\tag{19}$$

4. Repairable parallel redundant units with different failure rate and repair rate

In the last section, we discussed the non-repairable parallel redundant system with different failure rate. In this section we will discuss the repairable parallel redundant system with different failure rate and repair rate.

A state transition diagram is usually developed to illustrate the connecting link and mathematical relationship among each discrete state as shown in in Figure 4. The additional characteristic of this model of repairable parallel redundant system in Figure 4, compared with the model of the non-repairable parallel redundant system in Figure 3, is that when unit 1 or unit 2 has failed in the sequential state 2 or 3, a corrective maintenance will be commenced to repair the failed unit which is represented by the repair rate μ in the model.

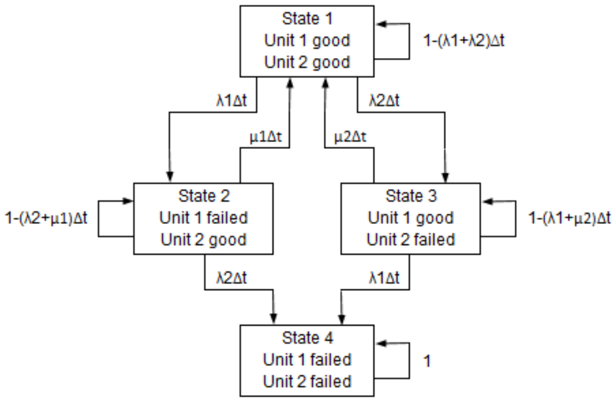

Figure 4: **Repairable system state transition diagram**

A transition matrix is developed to illustrate the mathematical function link among each sequential state in order to develop a set of first-order differential equations which describe the probability of being in each state. From the state diagram in Figure 4, we can write the transition matrix directly as given below:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{cccc} 1' & 2' & 3' & 4' \\ \end{array}$$

$$\begin{array}{c|cccc|} & 1' & 2' & 3' & 4' \\ 1 & 1-(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 \\ 2 & \mu_1 & 1-(\lambda_2+\mu_1) & 0 & \lambda_2 \\ 3 & \mu_2 & 0 & 1-(\lambda_1+\mu_2) & \lambda_1 \\ 4 & 0 & 0 & 0 & 1 \end{array}$$

From the transition matrix, we can write the probability equation of state two:

$$P_2(t+\Delta t) = P_1(t)\cdot\lambda_1\cdot\Delta t + P_2(t)\cdot\left[1-\left(\lambda_2+\mu_1\right)\cdot\Delta t\right] \tag{20}$$

Rearranging by moving P_2 (t) from the right-hand side to left-hand side, and dividing Δt on both sides of equation (20):

$$\frac{P_2(t+\Delta t)-P_2(t)}{\Delta t} = \frac{dP_2(t)}{dt} = \lambda_1\cdot P_1(t) - (\lambda_2+\mu_1)\cdot P_2(t) \tag{21}$$

By integrating equation (21), we obtain:

$$\int_0^\infty dP_2\left(t\right) = \lambda_1\cdot\int_0^\infty P_1\left(t\right)dt - (\lambda_2+\mu_1)\int_0^\infty P_2\left(t\right)dt \tag{22}$$

$$P_2\left(\infty\right) - P_2\left(0\right) = \lambda_1\cdot T_1 - \left(\lambda_2+\mu_1\right)\cdot T_2 \tag{23}$$

Where, the boundary condition:

$$P_2(\infty) = 0; P_2(0) = 0$$

Note that the boundary condition is equal to one at the state of $P_1(0)$ or $P_4(\infty)$, and zero at all other states.

T_1 is defined as the expected time in state one, and T_2 is the expected time in state two.

$$T_2 = \frac{\lambda_1}{\lambda_2 + \mu_1} T_1 \tag{24}$$

We can write the probability equation of state 3 using the same approach and obtain:

$$T_3 = \frac{\lambda_2}{\lambda_1 + \mu_2} T_1 \tag{25}$$

The probability equation of state four uses the same approach and obtain:

$$P_4(t + \Delta t) = \lambda_2 \cdot \Delta t \cdot P_2(t) + \lambda_1 \cdot \Delta t \cdot P_3(t) + P_4(t) \tag{26}$$

Rearranging by moving $P_4(t)$ from the right-hand side to left-hand side, and dividing Δt on the both sides of equation (26):

$$\frac{P_4(t + \Delta t) - P_4(t)}{\Delta t} = \frac{dP_4(t)}{dt} = \lambda_2 \cdot P_2(t) + \lambda_1 \cdot P_3(t) \tag{27}$$

By integrating equation (27), we obtain:

$$\int_0^\infty dP_4(t) = \lambda_2 \cdot \int_0^\infty P_2(t)\,dt + \lambda_1 \int_0^\infty P_3(t)\,dt \tag{28}$$

$$P_4(\infty) - P_4(0) = \lambda_2 \cdot T_2 + \lambda_1 \cdot T_3 \tag{29}$$

Where, the boundary condition:

$$P_4(\infty) = 1; P_4(0) = 0$$

Substituting the boundary condition into equation (29):

$$1 - 0 = \lambda_2 \cdot T_2 + \lambda_1 \cdot T_3 \tag{30}$$

We can substitute T_2 in equation (24) and T_3 in equation (25) directly into the above equation (30) and solve T_1 as follows:

$$1 = \frac{\lambda_1 \lambda_2}{\lambda_2 + \mu_1} T_1 + \frac{\lambda_1 \lambda_2}{\lambda_1 + \mu_2} T_1 \tag{31}$$

Solving T_1:

$$T_1 = \frac{\lambda_1\lambda_2\left(\lambda_1+\lambda_2+\mu_1+\mu_2\right)}{\left(\lambda_1+\mu_2\right)\left(\lambda_2+\mu_1\right)} \tag{32}$$

Here, the success of the system is defined by state one, state two and state three. State four is the failed condition. Consequently, we can write the MTBF. The MTBF would be defined as the sum of the expected time in state one, two and three. Mathematically, this can be written as:

$$MTBF = T_1+T_2+T_3 = \left(1+\frac{\lambda_1}{\lambda_2+\mu_1}+\frac{\lambda_2}{\lambda_1+\mu_2}\right)\cdot T_1 \tag{33}$$

Rearranging to obtain:

$$MTBF = \frac{\left(\lambda_1+\mu_2\right)\left(\lambda_2+\mu_1\right)+\lambda_1\left(\lambda_1+\mu_2\right)+\lambda_2\left(\lambda_2+\mu_1\right)}{\left(\lambda_1+\mu_2\right)\left(\lambda_2+\mu_1\right)}\cdot T_1 \tag{34}$$

Substituting T_1 from equation (32) into (34):

$$MTBF = \frac{\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2+\lambda_1\mu_1+\lambda_1\mu_2+\lambda_2\mu_1+\lambda_2\mu_2+\mu_1\mu_2}{\lambda_1\lambda_2\left(\lambda_1+\lambda_2+\mu_1+\mu_2\right)} \tag{35}$$

Note that when unit 1 and unit 2 have the same failure rate and repair rate, we have the following familiar equation (36):

$$MTBF = \frac{3\lambda+\mu}{2\lambda^2} \tag{36}$$

In practice, $(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)$ is much smaller than $\lambda_1\mu_1+\lambda_1\mu_2+\lambda_2\mu_1+\lambda_2\mu_2+\mu_1\mu_2$. Therefore, we could take $(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)$ out of the MTBF equation (35) and obtain the following approximation equation (37):

$$MTBF = \frac{\lambda_1\mu_1+\lambda_1\mu_2+\lambda_2\mu_1+\lambda_2\mu_2+\mu_1\mu_2}{\lambda_1\lambda_2(\lambda_1+\lambda_2+\mu_1+\mu_2)} = \frac{\mu_1\mu_2+(\lambda_1+\lambda_2)(\mu_1+\mu_2)}{\lambda_1\lambda_2(\lambda_1+\lambda_2+\mu_1+\mu_2)} \tag{37}$$

This MTBF equation (37) has the same result as equation 2 in Table 6.2-3, Reliability Toolkit: Commercial Practice Edition [4]:

$$\lambda_{1/2} = \frac{\lambda_A\lambda_B\left[\left(\mu_A+\mu_B\right)+\left(\lambda_A+\lambda_B\right)\right]}{\mu_A\mu_B+\left(\mu_A+\mu_B\right)\left(\lambda_A+\lambda_B\right)}$$

The difference between these two equations is that equation (37) calculates the expected time (MTBF), and equation 2 calculates the effective failure rate λ.

5. Conclusion

Markov Modeling is a useful and powerful modeling and analysis technique with applications in reliability analyses that are time-based. The reliability characteristics or behavior of a system is characterized using a state transition diagram, which consists of a set of discrete states that the system can be in, and defines the rate at which transitions between those states occur. As such, Markov models consist of possible chains of events that are representations of the sequences of failures, and in some cases repairs that are used to approximate the reliability and dependability of a system. The Markov model is analyzed to determine the probability of being in a given discrete state at a given time. By adding up the probabilities of discrete states that represent system success to determine the reliability and dependability of a system.

The specific equations derived in this paper (i.e. $MTTF = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$ for the non-repairable parallel redundant system with different failure rate, and $MTBF = \frac{\mu_1\mu_2 + (\lambda_1 + \lambda_2)(\mu_1 + \mu_2)}{\lambda_1\lambda_2\left(\lambda_1 + \lambda_2 + \mu_1 + \mu_2\right)}$ for the repairable parallel redundant system with different failure rate and repair rate) can be used in the common reliability approximation practice.

References

1. IEC 61165. Application of Markov techniques, Second edition 2006-06, Annex B, p.43-46.
2. MIL-HDBK-338B. Military Handbook Electronic Reliability Design Handbook, Notice 2, 11 May 2012, p. 367-373
3. RADC-TR-77-287 (DEC1977). A Redundancy Notebook, Rome Air Development Center, Air Force System Command, Griffiss Air Force Base, New York 13441, p. 22-35.
4. Reliability Toolkit: Commercial Practices Edition, Reliability Analysis Center, 6.2, p.161.